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Exercise 13 (Solution)

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Homework Problem 13.1.

Let U, H be Hilbert spaces and $S \in \mathcal{L}(U, H)$ as well as $\gamma \geqslant 0$ be a real number. Show Lemma 11.1 of the lecture notes, i. e., that the operator $S^\circ S + \gamma \operatorname{id}$ is self-adjoint and positive semi-definite and positive definite, if $\gamma > 0$.

Solution.

The operator $S^\circ S + \gamma \operatorname{id}$ maps U into U and for all $u, v \in U$, we have that

$$((S^\circ S+\gamma \operatorname{id})u, v)_U = (S^\circ Su, v)_U+\gamma(u, v)v = (Su, Sv)_H+\gamma(u, v)_U = (u, S^\circ Sv)_U+\gamma(u, v)_U = (u, (S^\circ S+\gamma \operatorname{id})v)_U.$$

Homework Problem 13.2.

Consider the quadratic objective

$$f(u) = \underbrace{\frac{1}{2}(A\,u, u)_U}_{\text{quadratic}} - \underbrace{(b, u)_U}_{\text{linear}} + \underbrace{c}_{\text{constant}} \tag{11.1}$$

from the lecture notes. Show Lemma 11.2 of the lecture notes, i. e., the identities:

$$f(u + \alpha\,d) = f(u) + \alpha\,\big(\underbrace{A\,u - b}_{=\nabla f(u)}, d\big)_U + \frac{\alpha^2}{2}(A\,d, d)_U \tag{11.4a}$$

$$\frac{\mathrm{d}}{\mathrm{d}\alpha} f(u + \alpha\,d) = \big(\nabla f(u + \alpha\,d), d\big)_U. \tag{11.4b}$$

Solution.

Because of self-adjointness of A, we have that

$$\begin{aligned} f(u+\alpha d) &= \frac{1}{2}(A\,(u+\alpha d), (u+\alpha d))_U - (b, (u+\alpha d))_U + c \\ &= \frac{1}{2}(A\,u, u)_U + \alpha (A\,u, d)_U + \frac{\alpha^2}{2}(d, d)_U - (b, u)_U - \alpha (b, d)_U + c \\ &= f(u) + \alpha (Au - b, d) + \frac{\alpha^2}{2}(d, d)_U. \end{aligned}$$

as well as

$$\frac{\mathrm{d}}{\mathrm{d}\alpha} f(u + \alpha\, d) = \alpha (d, d)_U + (Au - b, d) = (A(u + \alpha d) - b, d) = \big(\nabla f(u + \alpha\, d), d\big)_U.$$

You are not expected to turn in your solutions.