Exercise 12 (Solution)

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Homework Problem 12.1.

Consider the operator $T \colon L^2(0,1) \mapsto L^2(0,1)$ defined by

$$Tu(x) \coloneqq \int_0^x u(t) \,\mathrm{d}t \text{ for } x \in [0,1].$$

Find the adjoint of T with respect to the standard inner product.

Solution.

From Fubini's theorem, we obtain that

$$\begin{aligned}(Tu, v) &= \int_0^1 \int_0^x u(t) \,\mathrm{d}t v(x) \,\mathrm{d}x \\ &= \int_0^1 \int_0^x u(t) v(x) \,\mathrm{d}t \,\mathrm{d}x \\ &= \int_0^1 \int_t^1 u(t) v(x) \,\mathrm{d}x \,\mathrm{d}t \\ &= \int_0^1 u(t) \int_t^1 v(x) \,\mathrm{d}x \,\mathrm{d}t.\end{aligned}$$

Accordingly, the adjoint operator is given by $T^\circ(v)(x) = \int_x^1 v(t) \,\mathrm{d}t$.

Homework Problem 12.2. (homework problem 10.1 revisited)

Derive first order optimality conditions for the boundary-controlled modification of the floor-heating problem

$$\begin{aligned} \text{Minimize} \quad & \frac{1}{2}\|y - y_d\|^2_{L^2(\Omega_{\text{obs}})} + \frac{\gamma}{2}\|u\|^2_{L^2(\Gamma)} \\ \text{s. t.} \quad & \begin{cases} -\operatorname{div}\big(\kappa \nabla y\big) = 0 & \text{in } \Omega \\ \kappa \dfrac{\partial}{\partial n} y = \alpha\,(u - y) & \text{on } \Gamma \end{cases} \\ \text{and} \quad & u \in L^2(\Gamma). \end{aligned}$$

Solution.

First off, we can clearly apply Theorem 10.1 to the reduced (weak) formulation of the boundary control problem, as the result in the lecture notes is formulated on an abstract level and our boundary control problem is of the required form. Since we have $U_{\text{ad}} = L^2(\Gamma)$, i. e., an entire linear space, instead of a variational *in*equality, we end up with an equality. Specifically, if our reduced problem is of the form

$$\text{Minimize} \quad \frac{1}{2}\|\underbrace{E\,G}_{=:S}(u) - y_d\|^2_{L^2(\Omega)} + \frac{\gamma}{2}\|u\|^2_{L^2(\Gamma)} \quad \text{with} \quad u \in L^2(\Gamma)$$

with $E\colon H^1 \to L^2(\Omega)$, $G\colon L^2(\Gamma) \to H^1(\Omega)$, then the corresponding first order optimality condition is of the form

$$(S^\circ(S(u) - y_d), \delta u)_{L^2(\Gamma)} + \gamma(u, \delta u, =)0 \quad \text{for all } \delta u \in L^2(\Gamma),$$

i. e., by the isomorphism property of the Riesz operator,

$$S^\circ(S(u) - y_d) + \gamma u = 0 \text{ in } L^2(\Gamma).$$

As in the lecture, we now need to characterize the form of the adjoint solution operator S°. We can make the a guess from the following observations: The operator E is exactly the same as in the lecture. The operator G can be decomposed into two parts, the first being the mapping from $L^2(\Gamma)$ to $(H^1(\Omega))^*$, and the second being the solution operator of the PDE from an $(H^1(\Omega))^*$ element to a $H^1(\Omega)$ state. The first will have changed while the second will not have changed. The adjoint PDE in the lecture only has observation domain information on the right hand side, so we can expect the adjoint PDE not to change but the change to be present in how the adjoint state plys into the gradient information.

Let's see if the formal Lagrange method yields a more concrete guess. Our Lagrangian reads as

$$\mathcal{L}(y, u, p) \coloneqq \frac{1}{2}\|y - y_d\|^2_{L^2(\Omega)} + \frac{\gamma}{2}\|u\|^2_{L^2(\Gamma)} + \int_\Omega \kappa \nabla y \cdot \nabla p \, \mathrm{d}x + \int_\Gamma \alpha\, y\, p \, \mathrm{d}s - \int_\Gamma \alpha\, u\, p \, \mathrm{d}s.$$

The first equality in the stationarity condition is

$$\mathcal{L}_y(y, u, p)\,\delta y = \int_\Omega (y - y_d)\,\delta y \, \mathrm{d}x + \int_\Omega \kappa \nabla \delta y \cdot \nabla p \, \mathrm{d}x + \int_\Gamma \alpha\,\delta y\, p \, \mathrm{d}s = 0.$$

As expected, this is exactly the equation derived in the lecture for the domain controlled case. We know the third condition $\mathcal{L}_p(y, u, p)\,\delta p = 0$ for all $\delta p \in H^1(\Omega)$ is exactly the PDE constraint we started out with, so all the changes have to be in the second condition, namely the condition

$$\int_\Gamma (\gamma\, u - p)\,\delta u \;\mathrm{d}x = 0 \text{ for all } \delta u \in L^2(\Gamma),$$

which carries the implicit information that p is to be restricted to $p|_\Gamma = \tau(p)$ (i. e. in the sense of the trace operator, not a simple restriction operator for subdomains of Ω_{ctrl}).

Consequently, we expect the optimality system to be given for $u \in L^2(\Gamma)$, $y, p, v \in H^1(\Omega)$ by

$$\begin{aligned}
\int_\Omega \kappa \nabla p \cdot \nabla v + \int_\Gamma \alpha\, p\, v \;\mathrm{d}s &= \int_\Omega (y - y_d)\, v \;\mathrm{d}x && \text{for all } v \in H^1(\Omega) && \text{(Adjoint equation)}\\
\int_\Gamma \gamma\, u\, v \;\mathrm{d}s - \int_\Gamma \alpha\, p\, v \;\mathrm{d}s &= 0 && \text{for all } v \in L^2(\Gamma) && \text{(Derivative condition)}\\
\int_\Omega \kappa \nabla y \cdot \nabla v + \int_\Gamma \alpha\, y\, v \;\mathrm{d}s &= \int_\Gamma \alpha\, u\, v \;\mathrm{d}s && \text{for all } v \in H^1(\Omega) && \text{(State equation).}
\end{aligned}$$

Since the second and third line are a direct consequence of the definition of the solutoin operator and Theorem 10.1, we only need to check that in fact

$$S^\circ h = \tau(p) = p|_\Gamma$$

with p given as the solution of the adjoint equation, which has exactly the strong form givn in Equation (10.5) of the lecture notes with $\chi_{\text{obs}} = 1$. To that end, as in the lecture notes, we use the adjoint state p as a test function in the state equation and vice-versa to obtain that the left hand sides coincide and therefor the right hand sides do as well to yield

$$\int_\Omega h\, p \;\mathrm{d}x = \int_\Gamma \alpha\, u\, y \;\mathrm{d}s$$

which is exactly the defining equation of the adjoint operator.

You are not expected to turn in your solutions.