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Exercise 11

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Homework Problem 11.1.

(a) (i) Let X, Y, Z be normed linear spaces and $F\colon X \to Y$ and $G\colon Y \to Z$ be Fréchet differentiable at $x \in X$ and $F(x) \in Y$, respectively. Show that $G \circ F\colon X \to Z$ is Fréchet differentiable at x.

(ii) Give an example of normed linear spaces X, Y, Z and functions $F\colon X \to Y$ and $G\colon Y \to Z$, that are Gâteaux differentiable at $x \in X$ and $F(x) \in Y$, respectively, where $G \circ F$ is not Gâteaux-differentiable at x.

(b) Let $F\colon X \to Y$ be a function between two linear spaces X and Y, and let $\|\cdot\|_X$ and $\|\cdot\|_{X'}$ as well as $\|\cdot\|_Y$ and $\|\cdot\|_{Y'}$ be norms on X and Y, respectively. Further, let $x \in X$.

(i) Show that if F is Fréchet differentiable with respect to $\|\cdot\|_X$ and $\|\cdot\|_Y$, then it is Fréchet differentiable with respect to $\|\cdot\|_{X'}$ and $\|\cdot\|_{Y'}$, if $\|\cdot\|_{X'}$ is stronger than $\|\cdot\|_X$ and $\|\cdot\|_{Y'}$ is weaker than $\|\cdot\|_Y$.

(ii) Show that the operator $u \mapsto \sin(u)$ is Fréchet-differentiable as an operator from $L^{p_1}(0,1)$ to $L^{p_2}(0,1)$ for $p_1, p_2 \in [1, \infty)$ if and only if $p_2 < p_1$.

Hint: Taylor's theorem will be helpful. In the proof of differentiability, consider $q \geqslant \frac{p}{2}$. In the counter example, consider step functions h and $u = 0$.

You are not expected to turn in your solutions.