Exercise 11 (Solution)

Date issued: 7th January 2025

Homework Problem 11.1.

(a) (i) Let X, Y, Z be normed linear spaces and $F \colon X \to Y$ and $G \colon Y \to Z$ be Fréchet differentiable at $x \in X$ and $F(x) \in Y$, respectively. Show that $G \circ F \colon X \to Z$ is Fréchet differentiable at x.

(ii) Give an example of normed linear spaces X, Y, Z and functions $F \colon X \to Y$ and $G \colon Y \to Z$, that are Gâteaux differentiable at $x \in X$ and $F(x) \in Y$, respectively, where $G \circ F$ is not Gâteaux-differentiable at x.

(b) Let $F \colon X \to Y$ be a function between two linear spaces X and Y, and let $\|\cdot\|_X$ and $\|\cdot\|_{X'}$ as well as $\|\cdot\|_Y$ and $\|\cdot\|_{Y'}$ be norms on X and Y, respectively. Further, let $x \in X$.

(i) Show that if F is Fréchet differentiable with respect to $\|\cdot\|_X$ and $\|\cdot\|_Y$, then it is Fréchet differentiable with respect to $\|\cdot\|_{X'}$ and $\|\cdot\|_{Y'}$, if $\|\cdot\|_{X'}$ is stronger than $\|\cdot\|_X$ and $\|\cdot\|_{Y'}$ is weaker than $\|\cdot\|_Y$.

(ii) Show that the operator $u \mapsto \sin(u)$ is Fréchet-differentiable as an operator from $L^{p_1}(0,1)$ to $L^{p_2}(0,1)$ for $p_1, p_2 \in [1, \infty)$ if and only if $p_2 < p_1$.

Hint: Taylor's theorem will be helpful. In the proof of differentiability, consider $q \geqslant \frac{p}{2}$. In the counter example, consider step functions h and $u = 0$.

Solution.

(a) (i) We show that the derivative of $G \circ F$ at x is given by $G'(F(x))F'(x)$, as can be expected from a chain rule. To that end, note that for $h \in X$:

$$\begin{aligned}
&G(F(x+h)) - G(F(x)) - G'(F(x))F'(x)h \\
=&G(F(x) + F'(x)h + r_F(x,h)) - G(F(x)) - G'(F(x))F'(x)h \\
=&G'(F(x))r_F(x,h) + r_G(F(x), F'(x)h + r_F(x,h))
\end{aligned}$$

and therefore

$$\begin{aligned}
&\frac{\|G(F(x+h)) - G(F(x)) - G'(F(x))F'(x)h\|_Z}{\|h\|_X} \\
=&\frac{\|G'(F(x))r_F(x,h) + r_G(F(x), F'(x)h + r_F(x,h))\|_Z}{\|h\|_X} \\
\leqslant&\|G'(F(x))\|\frac{\|r_F(x,h)\|_Y}{\|h\|_X} + \frac{\|r_G(F(x), F'(x)h + r_F(x,h))\|_Z}{\|h\|_X} \\
=&\|G'(F(x))\|\frac{\|r_F(x,h)\|_Y}{\|h\|_X} + \frac{\|r_G(F(x), F'(x)h + r_F(x,h))\|_Z}{\|F'(x)h + r_F(x,h))\|_Y}\frac{\|F'(x)h + r_F(x,h))\|_Y}{\|h\|_X} \\
\leqslant&\|G'(F(x))\|\frac{\|r_F(x,h)\|_Y}{\|h\|_X} + \frac{\|r_G(F(x), F'(x)h + r_F(x,h))\|_Z}{\|F'(x)h + r_F(x,h))\|_Y}\|F'(x)\|\frac{\|h\|_X + \|r_F(x,h))\|_Y}{\|h\|_X}
\end{aligned}$$

with the right hand side term converging to 0 as h to 0 because of the two Fréchet differentiabilities assumed on the data.

(*ii*) Consider the functions $F\colon \mathbb{R}^2 \to \mathbb{R}$ and $G\colon \mathbb{R} \to \mathbb{R}^2$ given by

$$G(x_1, x_2) = \begin{cases} 1 & x_2 = x_1^2 \text{ and } x_1 > 0 \\ 0 & \text{else} \end{cases} \qquad F(x) = (x, x^2).$$

At $x = 0$, $F(x) = (0,0)$ with Fréchet derivative $(1,0)^\intercal$ and $G(F(x)) = G(0) = 0$ with Gâteaux derivative $G'(0) = 0$. However, $G \circ F$ is the Heaviside function that is not directionally differentiable at 0.

(b) (*i*) Simple use of the norm estimates gives us

$$\begin{aligned}
\frac{\|F(x+h) - F(x) - F'(x)h\|_{Y'}}{\|h\|_{X'}} &\leqslant C_Y \frac{\|F(x+h) - F(x) - F'(x)h\|_Y}{\|h\|_{X'}} \\
&\leqslant \frac{C_Y}{C_X}\frac{\|F(x+h) - F(x) - F'(x)h\|_Y}{\|h\|_X} \xrightarrow{\|h\|_{X'}\to 0} 0.
\end{aligned}$$

Note that $\|h\|_X \leqslant C_X \|h\|_{X'} \xrightarrow{\|h\|_{X'}\to 0} 0$.

(*ii*) The sin function is smooth as a function on the reals, any directional derivative of the superposition operator therefore needs to coincide with this derivative in a pointwise sense. From Taylor's theorem, we know that

$$\sin(u+h) = \sin(u) + \cos(u)\,h + r(u,h).$$

for all $u, h \in \mathbb{R}$. Applying this expansion at each point $x \in [0,1]$ for functions $u(x), h(x) \in L^{p_1}$, we therefore obtain

$$\sin(u(x) + h(x)) = \sin(u(x)) + \cos(u(x))\, h(x) + r(u(x), h(x)).$$

At the constant zero function $u = 0$ and for $h_\varepsilon \coloneqq \chi_{[0,\varepsilon]}$ and obtain that

$$r(0, h_\varepsilon(x)) = \sin(h_\varepsilon(x)) - h_\varepsilon(x) = Ch_\varepsilon(x) = C\chi_{[0,\varepsilon]}$$

for a constant $C \in \mathbb{R}$. Accordingly

$$\frac{\|\sin(0 + h_\varepsilon) - \sin(0) - \cos(0)h_\varepsilon\|_{L^{p_2}}}{\|h_\varepsilon\|_{L^{p_1}}} = \frac{\|r\|_{L^{p_2}}}{\|h_\varepsilon\|_{L^{p_1}}} = \frac{\|Ch_\varepsilon\|_{L^{p_2}}}{\|h_\varepsilon\|_{L^{p_1}}} = \frac{\|C\chi_{[0,\varepsilon]}\|_{L^{p_2}}}{\|\chi_{[0,\varepsilon]}\|_{L^{p_1}}} = c\varepsilon^{\frac{1}{p_2} - \frac{1}{p_1}}.$$

This shows the nondifferentiability at 0 for $p_2 \geqslant p_1$ because of non-zero-convergence with $\varepsilon \longrightarrow 0$.

When $p_2 < p_1$, the above is not a contradiction to Fréchet differentiability. Instead, we note that from Taylor's theorem, the boundedness of the first and second derivatives as well as the Lipschitz continuity of sin, we obtain that there is a $C > 0$ such that $|r(u(x), h(x))| \le C|h(x)|^2$ and $|r(u(x), h(x))| \le C|h(x)|$ for arbitrary $u, h \in L^{p_1}$. So if we additionally assume that $p_2 \geqslant \frac{p_1}{2}$, and therefore $1 < \frac{p_1}{p_2} \leqslant 2$, we therefore obtain that $|r(u(x), h(x))| \le C|h(x)|^{\frac{p_1}{p_2}}$. Accordingly, we have that

$$\|r(u,h)\|_{L^{p_2}}^{p^2} = \int_0^1 |r(u(x), h(x))|_2^p \,\mathrm{d}x \leqslant C^{p_2} \int_0^1 |h|^{p_1} \,\mathrm{d}x = C^{p_2} \|h\|_{L^{p_1}}^{p^1},$$

and therefore

$$\begin{aligned} \frac{\|\sin(u+h) - \sin(u) - \cos(u)h\|_{L^{p_2}}}{\|h\|_{L^{p_1}}} &= \frac{\|r(u,h)\|_{L^{p_2}}}{\|h\|_{L^{p_1}}} \\ &\leqslant c \frac{\|h\|_{L^{p_1}}^{\frac{p_1}{p_2}}}{\|h\|_{L^{p_1}}} \\ &= c\|h\|_{L^{p_1}}^{\frac{p_1}{p_2} - 1} \xrightarrow{\|h\|_{p_1} \to 0} 0. \end{aligned}$$

When $q < \frac{p}{2}$, the result follows because of part (i) of the exercise.

You are not expected to turn in your solutions.