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Exercise 10

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Homework Problem 10.1. (Modified heating problem)

Consider the modification of the floor heating problem (7.3) of the lecture notes:

$$\begin{aligned} &\text{Minimize} \quad \frac{1}{2}\|y - y_d\|^2_{L^2(\Omega_{\text{obs}})} + \frac{\gamma}{2}\|u\|^2_{L^2(\Gamma)} \\ &\text{s. t.} \quad \begin{cases} -\operatorname{div}\big(\kappa\,\nabla y\big) = 0 & \text{in } \Omega \\ \kappa\,\dfrac{\partial}{\partial n} y = \alpha\,\big(u - y\big) & \text{on } \Gamma \end{cases} \\ &\text{and} \quad u \in L^2(\Gamma). \end{aligned} \tag{0.1}$$

(a) Explain how the problem that is modeled by (0.1) differs from the one modeled by (7.3) of the lecture notes.

(b) Derive the weak formulation of the PDE constraint.

(c) Show existence of a bounded linear control-to-state map $G\colon L^2(\Gamma) \to H^1(\Omega)$ for the PDE constraint in variational formulation given Assumption 7.6.

(d) Show existence of a globally optimal control $u \in L^2(\Gamma)$ provided Assumption 7.6 holds and $\gamma > 0$.

(e) Explain how additional bound constraints on the control or a nonzero right hand side in the PDE change the results presented above.

Homework Problem 10.2. (Differentiability results for operators)

(a) Let $\Omega \subseteq \mathbb{R}^n$ be an open and bounded set. Compute the directional derivative of the 1-norm $\|\cdot\|_1\colon L^1(\Omega) \to \mathbb{R}$. When is the map Gâteaux differentiable?

(b) Give examples of Banach spaces X, Y and operators $F\colon X \to Y$ such that at a point $x \in X$:

(i) every directional derivative of F exists but F is not Gâteaux-differentiable,

(ii) F is Gâteaux differentiable but not Fréchet differentiable.

You are not expected to turn in your solutions.