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Exercise 10 (Solution)

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Homework Problem 10.1. (Modified heating problem)

Consider the modification of the floor heating problem (7.3) of the lecture notes:

$$\begin{aligned}
&\text{Minimize} && \frac{1}{2}\|y - y_d\|^2_{L^2(\Omega_{\text{obs}})} + \frac{\gamma}{2}\|u\|^2_{L^2(\Gamma)} \\
&\text{s. t.} && \begin{cases} -\operatorname{div}\big(\kappa\,\nabla y\big) = 0 & \text{in } \Omega \\ \kappa\,\dfrac{\partial}{\partial n} y = \alpha\,\big(u - y\big) & \text{on } \Gamma \end{cases} \\
&\text{and} && u \in L^2(\Gamma).
\end{aligned} \tag{0.1}$$

(a) Explain how the problem that is modeled by (0.1) differs from the one modeled by (7.3) of the lecture notes.

(b) Derive the weak formulation of the PDE constraint.

(c) Show existence of a bounded linear control-to-state map $G\colon L^2(\Gamma) \to H^1(\Omega)$ for the PDE constraint in variational formulation given Assumption 7.6.

(d) Show existence of a globally optimal control $u \in L^2(\Gamma)$ provided Assumption 7.6 holds and $\gamma > 0$.

(e) Explain how additional bound constraints on the control or a nonzero right hand side in the PDE change the results presented above.

Solution.

(a) Instead of on the right hand side of the differential equation on Ω, the control enters as a boundary control in the Robin-boundary conditions. Specifically, the control is the prescribed temperature on the outside of the domain. This means that the control is much less direct, as all modifications to the state on the interior of Ω will have happened through dissipativity.

(b) By testing with sufficiently regular test functions and partial integration, we obtain that

$$\begin{aligned} 0 = -\int_\Omega \operatorname{div}(\kappa \nabla y) v &= \int_\Omega (\kappa \nabla y) \cdot \nabla v - \int_\Gamma \kappa \frac{\partial y}{\partial n} v \, \mathrm{d}s \\ &= \int_\Omega \kappa \nabla y \cdot \nabla v - \int_\Gamma \alpha \left(u - y\right) v \, \mathrm{d}s, \end{aligned}$$

i. e. the weak form of the PDE constraint is to find a $y \in H^1(\Omega)$ such that

$$\underbrace{\int_\Omega \kappa \nabla y \cdot \nabla v + \int_\Gamma \alpha \, y \, v \, \mathrm{d}s}_{=:a(y,v)} = \underbrace{\int_\Gamma \alpha \, u \, v \, \mathrm{d}s}_{=:\langle l, v \rangle_{H^1(\Omega)}}$$

for all $v \in H^1(\Omega)$.

(c) We apply the Lax-Milgram lemma. First off, note that the definition of $l \in H^1(\Omega)^*$ is a well-definition, because the map is clearly linear in v and it is bounded as

$$\begin{aligned} \left|\int_\Gamma \alpha \, u \, v \, \mathrm{d}s\right| \leqslant \int_\Gamma |\alpha \, u \, v \, \mathrm{d}s| \leqslant \|\alpha\|_{L^\infty(\Gamma)} \int_\Gamma |u \, v \, \mathrm{d}s| &\leqslant \|\alpha\|_{L^\infty(\Gamma)} \|u\|_{L^2(\Gamma)} \|v\|_{L^2(\Gamma)} \\ &\leqslant \|\alpha\|_{L^\infty(\Gamma)} \|u\|_{L^2(\Gamma)} \|\tau\|_{H^1(\Omega) \to L^2(\Gamma)} \|v\|_{H^1(\Omega)}. \end{aligned}$$

Bilinearity of $a \colon H^1(\Omega) \times H^1(\Omega) \longrightarrow \mathbb{R}$ is apparent and we obtain boundedness from the boundedness of the trace operator and the Cauchy-Schwarz inequality, as

$$\begin{aligned} |a(y, v)| &\leqslant \|\kappa\|_{L^\infty(\Omega)} \|\nabla y\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)} + \|\alpha\|_{L^\infty(\Gamma)} \|y\|_{L^2(\Gamma)} \|v\|_{L^2(\Gamma)} \\ &\leqslant C_1 \|\kappa\|_{L^\infty(\Omega)} \|y\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)} + C_2 \|\alpha\|_{L^\infty(\Gamma)} \|y\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)} \leqslant C \|y\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)} \end{aligned}$$

for all $y, v \in H^1(\Omega)$. Ellipticity is a consequence of Friedrichs-inequality, where

$$a(y, y) = \int_\Omega \kappa \nabla y \cdot \nabla y \, \mathrm{d}x + \int_\Gamma \alpha y^2 \, \mathrm{d}s \geqslant C \|y\|^2_{H^1(\Omega)}.$$

The control-to-state operator is the composition of the solution operator and the operator mapping u to l and hence linear and bounded.

(d) As shown above, the control-to-state map G is well-defined so that the reduced form analogously to (**??**) is defined. The set U_{ad} is nonempty, closed and convex. Since $\gamma > 0$, we know that the cost functional is radially unbounded. The result now follows from the existence theorem **??** for linear-quadratic problems.

(e) Bound constraints yield boundedness of the levelsets of admissible controls, which makes it possible to drop the assumption of $\gamma > 0$. A nonzero right hand side in the PDE would result in an affine-linear solution operator.

Homework Problem 10.2. (Differentiability results for operators)

(a) Let $\Omega \subseteq \mathbb{R}^n$ be an open and bounded set. Compute the directional derivative of the 1-norm $\|\cdot\|_1 \colon L^1(\Omega) \to \mathbb{R}$. When is the map Gâteaux differentiable?

(b) Give examples of Banach spaces X, Y and operators $F \colon X \to Y$ such that at a point $x \in X$:

  (i) every directional derivative of F exists but F is not Gâteaux-differentiable,

  (ii) F is Gâteaux differentiable but not Fréchet differentiable.

Solution.

(a) For $u, \delta u \in L^1(\Omega)$ we have that

$$\lim_{t \searrow 0} \frac{\|u + t\delta u\|_1 - \|u\|_1}{t} = \lim_{t \searrow 0} \frac{1}{t} \int_\Omega |u(x) + t\delta u(x)| - |u(x)| \,\mathrm{d}x$$

and $u + t\delta u$ converges to u both in L^1 as well as pointwise almost everywhere in Ω. We can split up the domain and apply Lebesgue's dominated convergence theorem on the subdomains to obtain that

$$
\begin{aligned}
\lim_{t \searrow 0} \frac{1}{t} \int_\Omega |u(x) + t\delta u(x)| - |u(x)| \,\mathrm{d}x &= \lim_{t \searrow 0} \frac{1}{t} \int_{\{u(x)>0\}} |u(x) + t\delta u(x)| - |u(x)| \,\mathrm{d}x \\
&\quad + \frac{1}{t} \int_{\{u(x)<0\}} |u(x) + t\delta u(x)| - |u(x)| \,\mathrm{d}x \\
&\quad + \frac{1}{t} \int_{\{u(x)=0\}} |u(x) + t\delta u(x)| - |u(x)| \,\mathrm{d}x \\
&= \lim_{t \searrow 0} \int_{\{u(x)>0, u+td \geqslant 0\}} \delta u(x) \,\mathrm{d}x \\
&\quad + \frac{1}{t} \int_{\{u(x)>0, u+td<0\}} |u(x) + t\delta u(x)| - |u(x)| \,\mathrm{d}x \\
&\quad - \int_{\{u(x)<0, u+td \leqslant 0\}} \delta u(x) \,\mathrm{d}x \\
&\quad + \frac{1}{t} \int_{\{u(x)<0, u+td>0\}} |u(x) + t\delta u(x)| - |u(x)| \,\mathrm{d}x \\
&\quad + \int_{\{u(x)=0\}} |t\delta u(x)| \,\mathrm{d}x \\
&= \int_{\{u(x)>0\}} \delta u(x) \,\mathrm{d}x \\
&\quad - \int_{\{u(x)<0\}} \delta u(x) \,\mathrm{d}x \\
&\quad + \int_{\{u(x)=0\}} |\delta u(x)| \,\mathrm{d}x,
\end{aligned}
$$

where the two mixed-domain-integrals vanish due to Lebesgue's dominated convergence theorem because their integrands are bounded by $|\delta u|$, which is in L^1, and the integrand (accounting for the domains by characteristic functions) converge pointwise almost everywhere to 0. The functional

$$
L^1(\Omega) \ni \delta u \mapsto \int_{\{u(x)>0\}} \delta u(x) \,\mathrm{d}x - \int_{\{u(x)<0\}} \delta u(x) \,\mathrm{d}x + \int_{\{u(x)=0\}} |\delta u(x)| \,\mathrm{d}x
$$

is clearly linear in δu if and only if $\{u = 0\}$ is a set of measure zero, in which case it is bounded by 1 in norm and therefore Gâteaux-differentiable.

(b) The easiest example to see is picking any infinite dimensional Banach space X and $Y = \mathbb{R}$. Then there exists a linear but unbounded operator $F \colon X \to Y$. Due to linearity, we of course have that for any $\delta x \in X$:

$$
\frac{F(x + t\delta x) - F(x)}{t} = \frac{F(x + t\delta x - x)}{t} = F(\delta x)
$$

so F yields its own directional derivative and hence the only operator that could be the Gâteaux derivative. As F is unbounded, it is clearly not Gâteaux differentiable.

The operator $x \mapsto F^2(x)$ is an example of a nonlinear operator, as the directional derivatives are always given by $\delta x \mapsto 2F(x)F(\delta x)$, which is unbounded at any x.

You are not expected to turn in your solutions.