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Exercise 9 (Solution)

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Homework Problem 9.1. (Traces in L^p)

Let $\Omega \coloneqq B_1^{\|\cdot\|_2}(0) \subseteq \mathbb{R}^2$. Show that there can not be an extension of the trace map $\tau\colon C(\overline{\Omega}) \to C(\partial\Omega)$ to a continous map on $L^2(\Omega)$.

Solution.

Consider the family of functions $\left(f^{(k)}\right) \in C(\Omega)$, where $f^{(k)}\colon x \mapsto \|x\|^k$. Then

$$\|f^{(k)}\|_{L^2}^2 = \int_\Omega \|x\|^k \,\mathrm{d}x = \int_0^{2\pi} \int_0^1 r^k \,\mathrm{d}r \,\mathrm{d}\varphi = 2\pi \frac{1}{k+1} \xrightarrow{k\to\infty} 0,$$

meaning that the $f^{(k)}$ converge to 0 in $L^2(\Omega)$, but as they are continuous, their boundary values on $\partial\Omega = S_2$ are 1 constantly, however the constant zero function is continuous as well with boundary values 0. This shows that the boundary trace operator can not be L^2-continuous (even on $C(\Omega)$). It really is the L^2 topology that does not work well with the continuity of the extension.

Homework Problem 9.2. (The Lax-Milgram lemma)

(a) Let $n \in \mathbb{N}$, $b \in \mathbb{R}^n$ and a symmetric $A \in R^{n\times n}$ such that $x^\intercal Ax > c\|x\|_2^2$ for a $c \in \mathbb{R}_>$. Use the Lax-Milgram lemma to show that the linear system $Ax = b$ has a unique solution $x \in \mathbb{R}^n$.

(b) Let H be a Hilbert space and let $A\colon H \mapsto H$ be a bounded, linear operator such that $(Ax, x) \geq 0$ for every $x \in H$. Use the Lax-Milgram lemma to show that the operator $\mathrm{id} + \alpha A\colon H \to H$ is bijective for every $\alpha \geqslant 0$. Show boundedness of A^{-1}.

Solution.

(a) We define the bilinear form $a\colon x \mapsto x^\top Ax$ on the Hilbert space $\mathbb{R}^2$ with the euclidean inner product. By assumption, we have that

$$a(x,x) = x^\top Ax > c\|x\|_2^2,$$

i. e., ellipticity of the bilinear form. Additionally, we have that

$$|a(x,y)| = |x^\top Ay| = (x, Ay)_2 \leqslant \|x\|_2\|Ay\|_2 \leqslant \|x\|_2\|A\|_{2\mapsto 2}\|y\|_2$$

where $\|A\|_{2\mapsto 2}$ denotes the operator norm, so a is bounded. By the Lax-Milgram theorem, we obtain existence and uniqueness of the solution (and even the estimate $\|x\|_2 \leqslant \frac{1}{c}\|b\|_2$.

(b) For injectivity, let $x, y \in H$ with $(\mathrm{id} + \alpha A)x = (\mathrm{id} + \alpha A)y$, then

$$\begin{aligned} 0 &= ((\mathrm{id} + \alpha A)(x - y), x - y) \\ &= (\mathrm{id}(x - y), x - y) + \alpha(A(x - y), x - y) \\ &= \|x - y\|^2 + \alpha(A(x - y), x - y) \\ &\geqslant (1 + \alpha)\|x - y\|^2 \geqslant 0 \end{aligned}$$

meaning that $x = y$.

For surjectivity, let $v \in H$ and consider its image under the Riesz map, i. e., $(v, \cdot) \in H^*$. As id and A are linear an bounded, their (scaled) sum is as well. Accordingly, we have the boundedness of the bilinear form $(x, y) \mapsto ((\mathrm{id} + \alpha A)x, x)$ because

$$((\mathrm{id} + \alpha A)x, y) \leqslant \|(\mathrm{id} + \alpha A)x\|\|y\| \leqslant \|\mathrm{id} + \alpha A\|\|x\|\|y\|.$$

Additionally, by the same computation as for the injectivity, we obtain ellipticity with the constant $1 + \alpha \geqslant 0$. Lax-Milgram's lemma therefore yields a unique x such that $(Ax, \cdot) = (v, \cdot)$ and invertability of the Riesz map yields $Ax = v$ uniquely.

That A^{-1} is linear is clear, boundedness follows from

$$\|A^{-1}v\| = \|x\| \leq \frac{1}{1 + \alpha}\|v\|$$

by the a-priori estimate in Lax-Milgram's lemma.

You are not expected to turn in your solutions.