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Exercise 8

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Homework Problem 8.1. (Compact operators in the direct method of variational calculus)

Let X, Y and Z be Banach spaces. A linear operator $A\colon X \to Y$ is called **compact** if it maps bounded sets to sets whose closure is compact.

(a) Show that $A\colon X \to Y$ is compact if and only if the sequence of images $A(x^{(k)})$ in Y for any a bounded sequence $x^{(k)}$ in X has a convergent subsequence.

(b) Show that if $A\colon X \to Y$ is compact, then A is continuous.

(c) Show that if A is compact, then for any $B \in \mathcal{L}(Y, Z)$ the operators $B \circ A$ and $A \circ B$ are compact.

(d) Explain how compactness of an operator can play a role in the proof of the existence of optimizers for optimization problems of the type (5.8) when applying the direct method of variational calculus.

Homework Problem 8.2. (Optimizer invariance for control-reduced problems)

Suppose that Y and U are normed linear spaces. Show Lemma 6.1, i. e., the following statements:

(a) Suppose that $G\colon U_{\text{ad}} \to Y$ provides, for any $u \in U_{\text{ad}}$, the unique solution $y = G(u)$ of the constraint $e(y, u) = 0$.

  (i) If (y^*, u^*) is a global minimizer of the original (6.1), then u^* is a global minimizer of the reduced problem (6.2).

  (ii) If u^* is a global minimizer of the reduced problem (6.2), then $(G(u^*), u^*)$ is a global minimizer of the original problem (6.1).

(b) Suppose in addition that $G\colon U_{\text{ad}} \to Y$ is continuous on U_{ad}.

  (i) If (y^*, u^*) is a local minimizer of the original (6.1), then u^* is a local minimizer of the reduced problem (6.2).

(*ii*) If u^* is a local minimizer of the reduced problem (6.2), then $(G(u^*), u^*)$ is a local minimizer of the original problem (6.1).

You are not expected to turn in your solutions.