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Exercise 7

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Homework Problem 7.1. (Convergence principle)

Suppose that X is a normed linear space and that $\left(x^{(k)}\right)$ is a sequence in X. Show Lemma 5.9, i. e., the following statements:

(a) The following are equivalent:
   (*i*) $x^{(k)} \to x$.
   (*ii*) Every subsequence of $\left(x^{(k)}\right)$ contains a subsequence that converges to x strongly.

(b) The following are equivalent:
   (*i*) $x^{(k)} \rightharpoonup x$.
   (*ii*) Every subsequence of $\left(x^{(k)}\right)$ contains a subsequence that converges to x weakly.

Homework Problem 7.2. (Characterization of weak sequential lower semi-continuity)

Suppose that X is a normed linear space and $f\colon X \to \mathbb{R}$ is a functional. Show Lemma 5.15, i. e., the equivalence of the following statements:

(a) f is weakly sequentially lower semi-continuous.
(b) The epigraph $\operatorname{epi} f$ is weakly sequentially closed.
(c) The sublevel sets $S_\alpha \coloneqq \left\{x \in X \,\middle|\, f(x) \leqslant \alpha\right\}$ are weakly sequentially closed (possibly empty) for all $\alpha \in \mathbb{R}$.

Homework Problem 7.3. (Hilbert spaces are reflexive)

Show Lemma 5.20, i. e., that Hilbert spaces are reflexive.

You are not expected to turn in your solutions.