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Exercise 7 (Solution)

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Homework Problem 7.1. (Convergence principle)

Suppose that X is a normed linear space and that $\big(x^{(k)}\big)$ is a sequence in X. Show Lemma 5.9, i. e., the following statements:

(a) The following are equivalent:

 (i) $x^{(k)} \to x$.

 (ii) Every subsequence of $\big(x^{(k)}\big)$ contains a subsequence that converges to x strongly.

(b) The following are equivalent:

 (i) $x^{(k)} \rightharpoonup x$.

 (ii) Every subsequence of $\big(x^{(k)}\big)$ contains a subsequence that converges to x weakly.

Solution.

(a) (i) $\Rightarrow$ (ii): Let $x^{(k)} \to x$ be a convergent sequenc and $x^{(k^{(l)})}$ be any subsequence. Then $x^{(k^{(l)})}$ itself converges to x, because for any $\varepsilon > 0$, there exists $k_0(\varepsilon) > 0$ such that $\|x^{(k)} - x\| \leqslant \varepsilon$ for all $k \geqslant k_0(\varepsilon)$, but by the definition of a subsequence, there exists an $l_0(k_0(\varepsilon))$ such that $k^{(l)} > k_0(\varepsilon)$ for all $l \geqslant l_0(k_0(\varepsilon))$.

(ii) $\Rightarrow$ (i): Suppose that $x^{(k)} \not\to x$, then there exists a $\varepsilon > 0$ and a subsequence $x^{(k^{(l)})}$ such that $\|x^{(k^{(l)})} - x\| \geqslant \varepsilon$. By ($ii$), this sequence would have a convergent subsequence, which is a contradiction.

(b) (i) $\Rightarrow$ (ii): Let $x^{(k)} \rightharpoonup x$ and $x^{(k^{(l)})}$ be any subsequence. Then $x^{(k^{(l)})}$ itself converges to x weakly, because for any $f \in X^*$, we have $\langle f, x^{(k)} - x\rangle \to 0$ strongly in $\mathbb{R}$ and applying statement (a) to this sequence yields $\langle f, x^{(k^{(l)})} - x\rangle \to 0$.

$(ii) \Rightarrow (i)$: Suppose that $x^{(k)} \not\rightharpoonup x$, i. e., there exists a $\hat{f} \in X^*$, such that $\langle \hat{f}, x^{(k)} - x \rangle \not\to 0$ in $\mathbb{R}$. Accordingly, there exists a subsequence $x^{(k^{(l)})}$ such that $|\langle \hat{f}, x^{(k^{(l)})} \rangle| \geqslant \varepsilon$, implying that $x^{(k^{(l)})}$ has no weakly convergent subsequence, and thus a contradiction.

Homework Problem 7.2. (Characterization of weak sequential lower semi-continuity)

Suppose that X is a normed linear space and $f\colon X \to \mathbb{R}$ is a functional. Show Lemma 5.15, i. e., the equivalence of the following statements:

(a) f is weakly sequentially lower semi-continuous.

(b) The epigraph $\operatorname{epi} f$ is weakly sequentially closed.

(c) The sublevel sets $S_\alpha \coloneqq \{x \in X \,|\, f(x) \leqslant \alpha\}$ are weakly sequentially closed (possibly empty) for all $\alpha \in \mathbb{R}$.

Solution.

(b) $\Rightarrow$ (a): Let $x^{(k)}$ be a sequence in X converging weakly to $x \in X$. Let $x^{(k^{(l)})}$ denote a subsequence where $f(x^{(k^{(l)})}) \to \liminf f(x^{(k)})$. Then $\left(x^{(k^{(l)})}, f\left(x^{(k^{(l)})}\right)\right)$ is in $\operatorname{epi} f$ and weakly convergent to $(x, \liminf f(x^{(k)}))$, which, by assumption, lies in $\operatorname{epi} f$ as well, meaning that $f(x) \leqslant \liminf f(x^{(k)})$.

(c) $\Rightarrow$ (b): Let $(x^{(k)}, \mu^{(k)})$ be a weakly convergent sequence in $\operatorname{epi} f$ with limit (x, μ). Then of course $x^{(k)}$ and $\mu^{(k)}$ are weakly convergent on their own and since $\mu^{(k)}$ is in $\mathbb{R}$, it is also strongly convergent to μ. Now, for every $\varepsilon > 0$, we can pass to subsequences $x^{(k^{(l)})}$ and $\mu^{(k^{(l)})}$ such that $\mu^{(k^{(l)})}$ and therefore $f(x^{(k^{(l)})})$ is bounded by $\mu + \varepsilon$. Accordingly, $x^{(k^{(l)})}$ is in the weakly sequentially closed sublevelset $S_{\mu+\varepsilon}$, so $x \in S_{\mu+\varepsilon}$ as well, i. e., $f(x) \leqslant \mu + \varepsilon$ for all $\varepsilon > 0$, so $f(x) \leqslant \mu$ and therefore $(x, \mu) \in \operatorname{epi} f$.

(a) $\Rightarrow$ (c): Let $x^{(k)} \rightharpoonup x$ and $x^{(k)} \in S_\alpha$, i. e., $f(x^{(k)}) \leqslant \alpha$. Then $f(x) \leqslant \liminf f(x^{(k)}) \leqslant \alpha$ and therefore $x \in S_\alpha$.

Homework Problem 7.3. (Hilbert spaces are reflexive)

Show Lemma 5.20, i. e., that Hilbert spaces are reflexive.

Solution.

We need to show that the canonical embedding $i\colon X \to X^{**}$ is surjective, i. e., that for every $x^{**} \in X^{**}$

there exists an $x \in X$, such that $\langle x^{**}, x^* \rangle_{X^*} = \langle i(x), x^* \rangle_{X^*} = \langle x^*, x \rangle_X$ for all $x^* \in X^*$. Since by theorem Theorem 4.14, both X and X^* are Hilbert spaces, there are corresponding Riesz-mappings $\Phi_X : X \to X^*$ and $\Phi_{X^*} : X^* \to X^{**}$. For any x^{**}, we obtain $i(\Phi_X^{-1}(\Phi_{X^*}^{-1}(x^{**}))) = x^{**}$, because

$$\begin{aligned} \langle i(\Phi_X^{-1}(\Phi_{X^*}^{-1}(x^{**}))), x^* \rangle_{X^*} &= \langle x^*, \Phi_X^{-1}(\Phi_{X^*}^{-1}(x^{**})) \rangle_X \\ &= (x^*, \Phi_{X^*}^{-1}(x^{**}))_{X^*} \\ &= (\Phi_{X^*}^{-1}(x^{**}), x^*)_{X^*} \\ &= \langle x^{**}, x^* \rangle_{X^*}. \end{aligned}$$

You are not expected to turn in your solutions.