Exercise 6

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Homework Problem 6.1. (Riesz's represetation theorem)

(a) Compute the Riesz-representatives of the following linear, bounded operators:

 (i) $\Phi\colon (L^2(0,1), (\cdot,\cdot)_{L^2}) \ni f \mapsto \int_0^{\frac{1}{2}} f(x)\,\mathrm{d}x \in \mathbb{R}$

 (ii) $\Phi\colon \left(L^2(0,1), (\cdot,\cdot)_{L^2_{1+x}}\right) \ni f \mapsto \int_0^1 f(x)\,\mathrm{d}x \in \mathbb{R}$ where $(\cdot,\cdot)_{L^2_{1+x}}$ is the weighted inner product $(f,g) \mapsto \int_0^1 (1+x) f(x) g(x)\,\mathrm{d}x$

(b) Let $C([-1,1])$ denote the space of continuous functions to be equipped with the inner product

$$(f,g) \mapsto \int_{-1}^{1} f(x)g(x)\,\mathrm{d}x.$$

Show that the mapping $\Phi\colon C([-1,1]) \ni f \mapsto f(0) \in \mathbb{R}$ is a linear functional and that there does not exist any $g \in C([-1,1])$ representing Φ with respect to the given inner product. Why is this not a contradiction of Riesz's representation theorem?

Homework Problem 6.2. (Box-bounded L^p functions)

Show that the set

$$A \coloneqq \{f \in L^p(\Omega) \mid a \leqslant f(x) \leqslant b \text{ for a.a. } x \in \Omega\} \tag{5.1}$$

is bounded and closed in L^p as stated in Example 5.3.

Hint: If $f^{(k)} \to f$ in L^1, then there is a subsequence $f^{(k^{(l)})}$ converging to f pointwise almost everywhere.

Homework Problem 6.3. (The weak topology)

Let $(V, \|\cdot\|_V)$ be a normed space. Show the following:

(a) The weak limit of a weakly-convergent sequence is unique.

Hint: You may apply the Hahn-Banach theorem.

(b) Norms equivalent to $\|\cdot\|_V$ induce the same weak topology.

(c) Show that if V is infinite-dimensional, then the weak topology is not induced by any norm.

Hint: You may use that in infinite dimensional spaces, weakly open sets are unbounded in the norm.

You are not expected to turn in your solutions.