R. Herzog, G. Müller
Universität Heidelberg

Exercise 6 (Solution)

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Homework Problem 6.1. (Riesz's represetation theorem)

(a) Compute the Riesz-representatives of the following linear, bounded operators:

(i) $\Phi\colon (L^2(0,1), (\cdot,\cdot)_{L^2}) \ni f \mapsto \int_0^{\frac{1}{2}} f(x)\,\mathrm{d}x \in \mathbb{R}$

(ii) $\Phi\colon \left(L^2(0,1), (\cdot,\cdot)_{L^2_{1+x}}\right) \ni f \mapsto \int_0^1 f(x)\,\mathrm{d}x \in \mathbb{R}$ where $(\cdot,\cdot)_{L^2_{1+x}}$ is the weighted inner product $(f,g) \mapsto \int_0^1 (1+x)f(x)g(x)\,\mathrm{d}x$

(b) Let $C([-1,1])$ denote the space of continuous functions to be equipped with the inner product

$$(f,g) \mapsto \int_{-1}^{1} f(x)g(x)\,\mathrm{d}x.$$

Show that the mapping $\Phi\colon C([-1,1]) \ni f \mapsto f(0) \in \mathbb{R}$ is a linear functional and that there does not exist any $g \in C([-1,1])$ representing Φ with respect to the given inner product. Why is this not a contradiction of Riesz's representation theorem?

Solution.

(a) (i) As

$$\int_0^{\frac{1}{2}} f(x)\,\mathrm{d}x = \int_0^1 \chi_{(0,\frac{1}{2})}(x)f(x)\,\mathrm{d}x = (\chi_{(0,\frac{1}{2})}, f)_{L^2}$$

for the characteristic function $\chi_{(0,\frac{1}{2})} \in L^\infty((0,1)) \subseteq L^2((0,1))$, clearly the representative is $\chi_{(0,\frac{1}{2})}$.

(ii) We are looking for a $g \in L^2(0,1)$, such that

$$\int_0^1 (1+x)f(x)g(x)\,\mathrm{d}x = \int_0^1 f(x)\,\mathrm{d}x,$$

which holds exactly if $g(x) = \frac{1}{1+x}$, which is an L^∞-function and therefore an L^2-function on $(0,1)$.

(b) Linearity is follows from the definition of the pointwise evaluation of the sum of linear functionals. If there were a Riesz representative for Φ, then, by Cauchy-Schwarz, the functional would be continuous (i. e. bounded). However, the functional is not continuous with respect to the L^2-inner product induced norm, as shown by the sequence of functions defined by

$$f_k(x) \coloneqq \begin{cases} k\sqrt{\frac{1}{k} - x} & x \in [0, \frac{1}{k}] \\ k\sqrt{x + \frac{1}{k}} & x \in [-\frac{1}{k}, 0] \\ 0 & \text{else} \end{cases},$$

which all have L^2-norm 1 but evaluate to k at 0. Therefore Φ is a member of the algebraic dual space, not the topological dual space, for which the Riesz representation theorem provides representatives. Additionally, in homework problem 3.2 we have seen, that this space is not L^1 complete, and similarly, we can show, that it is not L^2 complete, so it is an inner product space but not a Hilbert space, as required by the Ries representation theorem.

Homework Problem 6.2. (Box-bounded L^p functions)

Show that the set

$$A \coloneqq \{f \in L^p(\Omega) \mid a \leqslant f(x) \leqslant b \text{ for a.a. } x \in \Omega\} \tag{5.1}$$

is bounded and closed in L^p as stated in Example 5.3.

Hint: If $f^{(k)} \to f$ in L^1, then there is a subsequence $f^{(k^{(l)})}$ converging to f pointwise almost everywhere.

Solution.

Since the Ω in Example 5.3 is bounded, the essential boundedness of f in A by $\max(|a|, |b|)$ immediately yields L^p-boundedness of A due to the norm estimates in Lemma 2.25.

As for closedness, consider a L^p-convergent sequence of functions $f^{(k)}$ in A with limit f. By definition of A, they are pointwise almost everywhere bounded by a and b respectively. Since convergence in L^p implies L^1 convergence, we can extract $f^{(k^{(l)})}$ converging to f pointwise almost everywhere, meaning that f is pointwise almost everywhere bounded by a and b (from below and above) as well.

Homework Problem 6.3. (The weak topology)

Let $(V, \|\cdot\|_V)$ be a normed space. Show the following:

(a) The weak limit of a weakly-convergent sequence is unique.

Hint: You may apply the Hahn-Banach theorem.

(b) Norms equivalent to $\|\cdot\|_V$ induce the same weak topology.

(c) Show that if V is infinite-dimensional, then the weak topology is not induced by any norm.

Hint: You may use that in infinite dimensional spaces, weakly open sets are unbounded in the norm.

Solution.

(a) Let $x^{(k)} \rightharpoonup x$ and $x^{(k)} \rightharpoonup y$. By definition, $\langle f, x\rangle = \langle f, y\rangle$, i. e. $\langle f, x - y\rangle = 0$ for all $f \in V^*$. However, if $x \neq y$, then $x - y \neq 0$ and by Hahn-Banach, there exists a functional that is nonzero on $\{x - y\}$ yielding a contradiction, meaning that $x = y$.

(b) We consider an equivalent norm $\|\cdot\|$. Let U be $\|\cdot\|_V$-weakly-open, i. e., for every $x \in U$ there are ε, $n \in \mathbb{N}$ and $\|\cdot\|$-continuous linear functionals $f_1, \ldots, f_n$ such that

$$\left\{y \in V \,\middle|\, |\langle f_i, y - x\rangle| < \varepsilon \text{ for } i = 1, \ldots, n\right\} \subseteq U.$$

As implied by Lemma 2.12, the f_i are $\|\cdot\|$ continuous as well, immediately showing that U is $\|\cdot\|$-weakly-open as well and by symmetrie of norm equivalence, we obtain that the weak topologies of $\|\cdot\|_V$ and $\|\cdot\|$ coincide.

(c) Let V be infinite-dimensional. Assume that the weak topology is induced by a norm $\|\cdot\|$ on V. Then the open $\|\cdot\|$-balls $B_{\frac{1}{k}}(0)$ for $k \in \mathbb{N}$ are all $\|\cdot\|_V$-weakly-open sets. Therefore, the balls $B_{\frac{1}{k}}(0)$ are $\|\cdot\|_V$-unbounded. Accordingly, there exists a sequence $x^{(k)}$ where

$$\|x^{(k)}\| \leq \frac{1}{k} \quad \text{but} \quad \|x^{(k)}\|_V \geq k.$$

Now, since all $f \in V^*$ are continuous w.r.t. the weak topology, which is the same as the $\|\cdot\|$-topology, we also have that $x^{(k)}$ is weakly convergent to 0 and therefore bounded, giving a contradiction.

You are not expected to turn in your solutions.