R. Herzog, G. Müller
Universität Heidelberg

Exercise 5 (Solution)

Date issued: 11th November 2024

Homework Problem 5.1. (examples of operators and their norms)

Decide which of the following operators is a linear and bounded operator, and, if applicable, compute their respective operator norm. Here, Ω is assumed to be an open, bounded subset of $\mathbb{R}^n$ for an $n \in N$ and $[a, b]$ is a non-degenerated interval in $\mathbb{R}$ for $a, b \in \mathbb{R}$.

(a) $(L^2(\Omega), \|\cdot\|_{L^2}) \ni f \mapsto f \in (L^2(\Omega), \|\cdot\|_{L^2})$
(b) $(L^2(\Omega), \|\cdot\|_{L^2}) \ni f \mapsto f \in (L^1(\Omega), \|\cdot\|_{L^1})$
(c) $(L^6(\Omega), \|\cdot\|_{L^6}) \ni f \mapsto f^3 \in (L^2(\Omega), \|\cdot\|_{L^2})$
(d) $(C([a, b]), \|\cdot\|_\infty) \ni f \mapsto f \cdot g \in (C([a, b]), \|\cdot\|_\infty)$ for a fixed $g \in C([a, b])$
(e) $(C([a, b]), \|\cdot\|_\infty) \ni f \mapsto f - \int_a^b f(t)\,\mathrm{d}t \in (C([a, b]), \|\cdot\|_\infty)$
(f) $(W^{1,2}(\Omega), \|\cdot\|_{W^{1,2}}) \ni f \mapsto f' \in (L^2(\Omega), \|\cdot\|_{L^2})$ (mapping every function to its weak derivative)

Solution.

All operators are linear, so we need to examine boundedness of them all.

(a) The identity map has operator norm 1, as $\|f\|_{L^2} \leqslant 1\,\|f\|_{L^2}$ for all $f \in L^2(\Omega)$, of course, and any f is an example showing that any constant lower than 1 is impossible.
(b) The Cauchy-Schwarz inequality yields that

$$\|f\|_{L^1} = \int_\Omega |f| 1\,\mathrm{d}x = (f, 1)_{L^2} \leqslant \|f\|_{L^2} \|1\|_2 = \sqrt{|\Omega|}\|f\|_{L^2}$$

showing that the operator is bounded and its norm is bounded by $\sqrt{|\Omega|}$, whereas the constant 1 function is an example showing, that this is in fact the norm, because

$$\|1\|_{L^1} = \int_\Omega 1 \,\mathrm{d}x = |\Omega| = \sqrt{|\Omega|}\sqrt{|\Omega|} = \sqrt{|\Omega|}\sqrt{\int_\Omega 1^2 \,\mathrm{d}x} = \sqrt{|\Omega|}\|1\|_{L^2}.$$

(c) We have that

$$\|f^3\|_{L^2} = \sqrt{\int_\Omega (f^3(x))^2 \,\mathrm{d}x} = \sqrt{\int_\Omega f^6(x) \,\mathrm{d}x} = (\|f\|_{L^6})^3$$

so the sequence of constant functions that map any input to $n \in \mathbb{N}$ shows that this is an unbounded operator.

(d) We have that

$$\|f \cdot g\|_\infty = \sup_{x \in [a,b]} f(x) \cdot g(x) \le \sup_{x \in [a,b]} f(x) \cdot \sup_{x \in [a,b]} g(x) = \|f\|_\infty \|g\|_\infty,$$

so the operator is bounded, and we actually obtain that its norm is $\|g\|_\infty$, as $f \equiv 1$ shows.

(e) The operator maps a continuous function to the same function shifted by its integral value over the same interval. We have that

$$\begin{aligned}\left\| f - \int_a^b f(s) \,\mathrm{d}s \right\|_\infty &= \sup_{x \in [a,b]} \left| f - \int_a^b f(s) \,\mathrm{d}s \right| \leqslant \sup_{x \in [a,b]} |f| + \left| \int_a^b f(s) \,\mathrm{d}s \right| \leqslant \|f\|_\infty + (b-a)\,\|f\|_\infty \\ &= (1 + (b-a))\,\|f\|_\infty,\end{aligned}$$

showing that the operator is in fact bounded. Its norm turns out to be the constant $1 + (b-a)$ seen above, because we can consider the family of continuous functions that are linear on $[a, a+\varepsilon]$ and constant on the remainder of the interval defined by

$$f_\varepsilon(x) \coloneqq \begin{cases} -1 + 2\frac{x-a}{\varepsilon} & x \in [a, a+\varepsilon] \\ 1 & x \in [a+\varepsilon, b] \end{cases},$$

whose integral values are $(b-a) - \varepsilon$ and whose minimum value (at a) is $1 + (b-a) - \varepsilon$.

This is an interesting example because here, there is no single continuous function where the supremum/infimum in the definition of the operator norm is actually attained.

(f) This is a bounded operator, because

$$\|f'\|_{L^2} = \sqrt{\|f'\|_{L^2}^2} \leqslant \sqrt{\|f'\|_{L^2}^2 + \|f\|_{L^2}^2} = \|f\|_{W^{1,2}}.$$

The norm is in fact 1, as for any f with f', we can shift the values of f up or down arbitrarily without changing f' to obtain an f with $\|f\|_{L^2} = 0$.

Homework Problem 5.2. (convergence in the operator norm implies pointwise convergence)

Suppose that X and Y are normed linear spaces and $\big(A^{(k)}\big)$ is a sequence of bounded linear operators $X \to Y$. Show Lemma 4.6 of the lecture notes, i. e., that if $A^{(k)}$ converges to $A \in \mathcal{L}(X, Y)$ in the operator norm, then $A^{(k)}(x)$ converges to $A(x)$ for all $x \in X$.

Solution.

Let $x \in X$ be given arbitrarily. Then

$$\|A^{(k)}(x) - A(x)\|_Y = \|A^{(k)} - A\|_{\mathcal{L}(X,Y)} \|x\|_X \xrightarrow{k\to\infty} 0.$$

Homework Problem 5.3. (boundedness is continuity)

Suppose that X and Y are normed linear spaces and $A\colon X \to Y$ is a linear operator. Prove Lemma 4.5 of the lecture notes, i. e., the equivalence of the following statements:

(a) A is continuous at 0.

(b) A is continuous on X.

(c) A is Lipschitz continuous.

(d) A is bounded.

Solution.

The fact that statement (c) $\Rightarrow$ statement (b) $\Rightarrow$ statement (a) is well known from basic analysis classes.

To show that statement (a) $\Rightarrow$ statement (d), let ε and δ be positive real numbers from the corresponding definition of continuity at 0 be given. Then for all $x \neq 0$, we have that

$$\|A(x)\|_Y = \frac{\|x\|_X}{\delta} \left\| A\left(\frac{\delta x}{\|x\|} - 0\right) \right\|_Y \le \frac{\|x\|_X}{\delta} \varepsilon = \underbrace{\frac{\varepsilon}{\delta}}_{=:c} \|x\|_X,$$

while obviously $A(0) = 0$ because of linearity.

Now, from boundedness (statement (d)), we show Lipschitz continuity (statement (c)) by choosing

x_1, x_2 from X, arbitrarily and obtain that

$$\|A(x_1) - A(x_2)\|_Y = \|A(x_1 - x_2)\|_Y = \underbrace{\|A\|_{\mathcal{L}(X,Y)}}_{=:L} \|x_1 - x_2\|_X.$$

Homework Problem 5.4. (composition of bounded linear operators)

Suppose that X, Y and Z are normed linear spaces and $B\colon Y \to Z$ as well as $A\colon X \to Y$ are bounded linear operators. Show that $B \circ A$ is a bounded linear operator from $X \to Z$ and show that

$$\|B \circ A\|_{\mathcal{L}(X,Z)} \leqslant \|B\|_{\mathcal{L}(Y,Z)} \|A\|_{\mathcal{L}(X,Y)}.$$

Give an example each for when this estimate holds with equality or strict inequality.

Solution.

Linearity of the composition is a standard linear algebra result. Since A and B are both bounded, we immediately obtain boundedness of $B \circ A$ because of

$$\|(B \circ A)(x)\|_Z = \|B(A(x))\|_Z \leqslant \|B\|_{\mathcal{L}(Y,Z)} \|A(x)\|_Y \leqslant \|B\|_{\mathcal{L}(Y,Z)} \|A\|_{\mathcal{L}(X,Y)} \|x\|_X$$

and accordingly the estimate $\|B \circ A\|_{\mathcal{L}(X,Z)} \leqslant \|B\|_{\mathcal{L}(Y,Z)} \|A\|_{\mathcal{L}(X,Y)}$.

This inequality does not necessarily holds with equality, but can. An example, where equality holds, is if one of the two operators is the zero map. An example, where a strict inequality holds is linear operators on $X = Y = Z = \mathbb{R}^2$ defined by the matrices

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \quad BA = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix},$$

which alle have norm 2.

Equality holds when considering the setting where A is applied twice, i. e., A^2.

You are not expected to turn in your solutions.