R. Herzog, G. Müller
Universität Heidelberg

Exercise 4

Date issued: 4th November 2024

Homework Problem 4.1. (Weak derivatives)

(a) Prove the statements in Example 2.35 of the lecture notes, i. e., that the function $f\colon (-1,1) \longrightarrow \mathbb{R}$ defined by $f(x) = |x|$ has the weak first-order derivative

$$w(x) = \begin{cases} -1 & \text{if } x < 0, \\ 1 & \text{if } x > 0, \end{cases}$$

but does not have a weak second-order derivative in $L^1_{\text{loc}}(-1,1)$.

(b) (Updated exercise statement)[GM] Let $\Omega \subseteq \mathbb{R}^n$ be a non-empty, bounded, open set, $i \in \{1,\ldots,n\}$ be given and let $x = (x_1,\ldots,x_n)$ be in Ω with $a_i, b_i \in \mathbb{R}$, such that

$$x \in B \coloneqq \{(x_1,\ldots,x_{i-1},y,x_{i+1},\ldots,x_n) \mid y \in (a_i,b_i)\} \subseteq \Omega.$$

Show that every class of functions in $W^{1,p}(\Omega)$ for $p \in [1,\infty)$ has a representative that is absolutely continuous on B as a function of in the i-th component.

(c) Let $\Omega = (0,1)^2$ and $f(x_1,x_2) = f_1(x_1) + f_2(x_2)$ where f_1, f_2 are in $L^1(0,1)$ but not absolutely continuous. Show that f does not have weak derivatives of first order order, but the second derivative for the multiindex $\alpha = (1,1)$ exists.

Homework Problem 4.2. (Inner products and Sobolev spaces)

(a) Suppose that $(V,(\cdot,\cdot))$ is an inner product space. Show Lemma 3.2, i. e., that

$$\|u\| \coloneqq \sqrt{(u,u)} \tag{3.2}$$

for $u \in V$ defines a norm on V.

(b) Prove the statement of Example 3.4 (i), i. e., that on $\mathbb{R}^n$ for $n \in \mathbb{N}_{>0}$, the possible inner products are in a bijective one-to-one correspondence with the symmetric[GM] positive definite matrices in $\mathbb{R}^{n \times n}$.

(c) Let $\Omega \subseteq \mathbb{R}^n$ be a non-empty, bounded, open set and let $\omega \in L^\infty(\Omega)$. Show that if there is a constant $c \in \mathbb{R}$, such that $0 < c \leqslant \omega$ almost everywhere in Ω, then

$$(u, v) \mapsto \sum_{|\alpha| \leqslant 1} \int_\Omega \omega \, D^\alpha u \, D^\alpha v \, \mathrm{d}x^{\mathrm{GM}}$$

defines a inner product on $W^{1,2}(\Omega)^{\mathrm{GM}}$ that induces an equivalent norm to $\|\cdot\|_{W^{1,2}(\Omega)}{}^{\mathrm{GM}}$ (as defined in (2.21)).

Why is ω's boundedness away from zero essential in this result?

You are not expected to turn in your solutions.