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Exercise 4 (Solution)

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Homework Problem 4.1. (Weak derivatives)

(a) Prove the statements in Example 2.35 of the lecture notes, i. e., that the function $f\colon (-1,1) \to \mathbb{R}$ defined by $f(x) = |x|$ has the weak first-order derivative

$$w(x) = \begin{cases} -1 & \text{if } x < 0, \\ 1 & \text{if } x > 0, \end{cases}$$

but does not have a weak second-order derivative in $L^1_{\text{loc}}(-1,1)$.

(b) (Updated exercise statement)[GM] Let $\Omega \subseteq \mathbb{R}^n$ be a non-empty, bounded, open set, $i \in \{1, \ldots, n\}$ be given and let $x = (x_1, \ldots, x_n)$ be in Ω with $a_i, b_i \in \mathbb{R}$, such that

$$x \in B \coloneqq \{(x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_n) \mid y \in (a_i, b_i)\} \subseteq \Omega.$$

Show that every class of functions in $W^{1,p}(\Omega)$ for $p \in [1, \infty)$ has a representative that is absolutely continuous on B as a function of in the i-th component.

(c) Let $\Omega = (0,1)^2$ and $f(x_1, x_2) = f_1(x_1) + f_2(x_2)$ where f_1, f_2 are in $L^1(0,1)$ but not absolutely continuous. Show that f does not have weak derivatives of first order order, but the second derivative for the multiindex $\alpha = (1,1)$ exists.

Solution.

(a) For the check of the first order derivative, note that $w(x)$ is essentially bounded by 1 and hence in $L^1_{\text{loc}}(\Omega)$. Now, let $\varphi \in C_c^\infty(-1,1)$, then, using integration by parts for a domain split at 0, we obtain that

$$\begin{aligned}\int_{-1}^{1} f\varphi'(x)\,\mathrm{d}x &= \int_{-1}^{1}|x|\varphi'(x)\,\mathrm{d}x = \int_{-1}^{0} -x\varphi'(x)\,\mathrm{d}x + \int_{0}^{1} x\varphi'(x)\,\mathrm{d}x \\ &= \int_{-1}^{0}\varphi(x)\,\mathrm{d}x + \underbrace{\varphi(-1)}_{0} + \int_{0}^{1}\varphi(x)\,\mathrm{d}x - \underbrace{\varphi(1)}_{0} = -\int_{-1}^{1} w(x)\varphi(x)\,\mathrm{d}x.\end{aligned}$$

Assuming, that there were a weak second derivative $v \in L^1_{\mathrm{loc}}(-1,1)$ of $|x|$, then for any $\varphi \in C_c^\infty(-1,1)$, we would have that

$$\begin{aligned}\int_{-1}^{1} v\varphi(x)\,\mathrm{d}x = \int_{-1}^{1} f\varphi''(x)\,\mathrm{d}x &= -\int_{-1}^{1} w(x)\varphi'(x)\,\mathrm{d}x = -\int_{-1}^{1} w(x)\varphi'(x)\,\mathrm{d}x \\ &= -\int_{-1}^{0} w(x)\varphi'(x)\,\mathrm{d}x - \int_{0}^{1} w(x)\varphi'(x)\,\mathrm{d}x = 2\varphi(0).\end{aligned}$$

Testing with all functions $\varphi \in C_c^\infty(-1,1)$ whose support lies in $(0,1)$ or $(-1,0)$, respectively, yields that $v = 0$ almost everywhere in $(-1,1)$, but this yields a contradiction when testing with functions $\varphi \in C_c^\infty(-1,1)$ with $\varphi(0) \neq 0$.

(b) Let $[f] \in W^{1,p}(\Omega)$. By definition, the first derivative of f with respect to the i-th component exists and is p-integrable an therefore integrable, so for any $y \in (a_i, b_i)$, we can define

$$g(y) \coloneqq \int_{a_i}^{y} \partial_i f(x_1, \dots, x_{i-1}, t, x_{i+1}, \dots, x_n)\,\mathrm{d}t.$$

This $g(y)$ is absolutely continuous because of the integral definition and the integrability of the integrand, which is a standard result from the theory of the lebesgue integral that can be proven using dominated convergence or the definition via simple functions.

Additionally, the weak derivative of g is $\partial_i f$ as a function of the i-th component, because for any $\varphi \in C_c^\infty(a_i, b_i)$, we have that

$$\begin{aligned}\int_{a_i}^{b_i} g(x)\varphi'(x)\,\mathrm{d}x &= \int_{a_i}^{b_i}\int_{a_i}^{x} \partial_i f(x_1, \dots, x_{i-1}, t, x_{i+1}, \dots, x_n)\varphi'(x)\,\mathrm{d}t\,\mathrm{d}x \\ &= \int_{a_i}^{b_i}\int_{t}^{b_i} \partial_i f(x_1, \dots, x_{i-1}, t, x_{i+1}, \dots, x_n)\varphi'(x)\,\mathrm{d}x\,\mathrm{d}t \\ &= \int_{a_i}^{b_i} \partial_i f(x_1, \dots, x_{i-1}, t, x_{i+1}, \dots, x_n)\,(\varphi(b_i) - \varphi(t))\,\mathrm{d}t \\ &= -\int_{a_i}^{b_i} \partial_i f(x_1, \dots, x_{i-1}, t, x_{i+1}, \dots, x_n)\varphi(t)\,\mathrm{d}t.\end{aligned}$$

Accordingly, we obtain that

$$\int_{a_i}^{b_i} (f(x_1, \dots, x_{i-1}, x, x_{i+1}, \dots, x_n) - g(x))\varphi' \, \mathrm{d}x = -\int_{a_i}^{b_i} (\partial_i f(x_1, \dots, x_{i-1}, x, x_{i+1}, \dots, x_n) - g'(x))\varphi \, \mathrm{d}x \\ = 0 \tag{0.1}$$

for every $\varphi \in C_c^\infty(a_i, b_i)$, meaning that there is a constant $c \in \mathbb{R}$ such that $f(x_1, \dots, x_{i-1}, x, x_{i+1}, \dots, x_n) - g(x) = c$, given any test function $\psi \in C_c^\infty(a_i, b_i)$ with $\int_{a_i}^{b_i} \psi \, \mathrm{d}x = 1$ and any other test function $\varphi \in C_c^\infty(a_i, b_i)$, we can construct

$$\varphi - \psi \int_{a_i}^{b_i} \varphi(s) \, \mathrm{d}s \in C^\infty(a_i, b_i)$$

whose integral vanishes, so

$$\Phi(x) \coloneqq \int_{a_i}^{x} \varphi - \psi \int_{a_i}^{b_i} \varphi(s) \, \mathrm{d}s \, \mathrm{d}t$$

is a test function in $C_c^\infty(a_i, b_i)$ and by (0.1), we obtain that

$$\begin{aligned} 0 &= \int_{a_i}^{b_i} (f(x_1, \dots, x_{i-1}, x, x_{i+1}, \dots, x_n) - g(x))\Phi'(x) \, \mathrm{d}x \\ &= \int_{a_i}^{b_i} (f(x_1, \dots, x_{i-1}, x, x_{i+1}, \dots, x_n) - g(x)) \left(\varphi - \psi \int_{a_i}^{b_i} \varphi(s) \, \mathrm{d}s\right) \mathrm{d}x \end{aligned}$$

which means that

$$\int_{a_i}^{b_i} (f(\dots, x, \dots) - g(x))\varphi \, \mathrm{d}x = \int_{a_i}^{b_i} \varphi(s) \, \mathrm{d}s \underbrace{\int_{a_i}^{b_i} (f(\dots, x, \dots) - g(x))\psi(x) \, \mathrm{d}x}_{=:c} \quad \forall \varphi \in C_c^\infty(a_i, b_i)$$

implying that $f(x_1, \dots, x_{i-1}, x, x_{i+1}, \dots, x_n) = c + g(x)$ for almost every $x \in (a_i, b_i)$, so $c + g(x)$ is an absolutely continuous representative of $[f]$.

(c) An example of functions f_1 and f_2 that are L^1 but not absolutely continuous is the function $x \mapsto \sin(\frac{1}{x})$, which is not even uniformly continuous.

The partial derivatives of first order can not exist, because, as seen in the last exercise, the function f is not but would have to be absolutely continuous in each variable. However, the weak mixed derivative for $\alpha = (1,1)$ is the zero function, as

$$\begin{aligned}\int_{(0,1)^2} f(x)\partial_{12}\varphi(x)\,\mathrm{d}x &= \int_{(0,1)^2} (f_1(x_1) + f_2(x_2))\,\partial_{12}\varphi(x)\,\mathrm{d}x \\ &= \int_{(0,1)^2} f_1(x_1)\partial_{12}\varphi(x)\,\mathrm{d}x + \int_{(0,1)^2} f_2(x_2)\partial_{12}\varphi(x)\,\mathrm{d}x \\ &= \int_{(0,1)} f_1(x_1) \underbrace{\int_{(0,1)} \partial_{12}\varphi(x)\,\mathrm{d}x_2}_{=0} \mathrm{d}x_1 + \int_{(0,1)} f_2(x_2) \underbrace{\int_{(0,1)} \partial_{12}\varphi(x)\,\mathrm{d}x_1}_{=0} \mathrm{d}x_2 \\ &= 0.\end{aligned}$$

Homework Problem 4.2. (Inner products and Sobolev spaces)

(a) Suppose that $(V, (\cdot,\cdot))$ is an inner product space. Show Lemma 3.2, i. e., that

$$\|u\| \coloneqq \sqrt{(u,u)} \tag{3.2}$$

for $u \in V$ defines a norm on V.

(b) Prove the statement of Example 3.4 (i), i. e., that on $\mathbb{R}^n$ for $n \in \mathbb{N}_{>0}$, the possible inner products are in a bijective one-to-one correspondence with the symmetric[GM] positive definite matrices in $\mathbb{R}^{n\times n}$.

(c) Let $\Omega \subseteq \mathbb{R}^n$ be a non-empty, bounded, open set and let $\omega \in L^\infty(\Omega)$. Show that if there is a constant $c \in \mathbb{R}$, such that $0 < c \leqslant \omega$ almost everywhere in Ω, then

$$(u,v) \mapsto \sum_{|\alpha|\leqslant 1} \int_\Omega \omega\, D^\alpha u\, D^\alpha v\,\mathrm{d}x^{\mathrm{GM}}$$

defines a inner product on $W^{1,2}(\Omega)^{\mathrm{GM}}$ that induces an equivalent norm to $\|\cdot\|_{W^{1,2}(\Omega)}{}^{\mathrm{GM}}$ (as defined in (2.21)).

Why is ω's boundedness away from zero essential in this result?

Solution.

(a) We simply check the defining properties of a norm. Well definedness and positive definiteness are a direct consequence of the positive definiteness of the inner product.

As for positive homogeniety, we have that for all $\alpha \in \mathbb{R}$,

$$\sqrt{(\alpha u, \alpha u)} = \sqrt{\alpha^2 (u,u)} = |\alpha| \sqrt{(u,u)}.$$

The triangle inequality holds because for $u, v \in V$, we have that

$$\begin{aligned}(u+v, u+v) &= (u,u) + 2(u,v) + (v,v) \\ &\leqslant (u,u) + 2\sqrt{(u,u)}\sqrt{(v,v)} + (v,v) \\ &= \sqrt{(u,u)}^2 + 2\sqrt{(u,u)}\sqrt{(v,v)} + \sqrt{(v,v)}^2 = (\sqrt{(u,u)} + \sqrt{(v,v)})^2\end{aligned}$$

due to the Cauchy-Schwarz-inequality (note that the proof only uses properties of the scalar product, not of the induced norm used in its statement in the lecture notes), and because of monotonicity of the square root function on the reals.

(b) For any symmetric positive definite Matrix $M \in \mathbb{R}^{n \times n}$, the map

$$(u,v)_M = u^\intercal M v$$

is in inner product.

Linearity is a consequence of the linearity of matrix-vector and matrix-matrix products. Symmetry is a consequence of the symmetry of M, where

$$(u,v)_M = u^\intercal M v = \left(\left(u^\intercal M v\right)^\intercal\right)^\intercal = \left(v^\intercal M^\intercal u\right)^\intercal = \left(v^\intercal M u\right) = (v,u)_M.$$

And finally, positive definiteness of the iner product follows from the matching property of M.

Now if M_1, M_2 define the same inner product, then

$$M_{1,i,j} = e_i^\intercal M_1 e_j = (e_i, e_j)_{M_1} = (e_i, e_j)_{M_2} = e_i^\intercal M_1 e_j = M_{2,i,j}$$

for all $i, j \in \{1, \ldots, N\}$, implying that $M_1 = M_2$.

(c) Well-definedness is due to the well-definedness of the standard $W^{1,2}$ inner product and the essential boundedness of the weight function ω. Symmetry and linearity are obvious. Positive definiteness follows, because

$$\int_\Omega \underbrace{\omega}_{\geqslant c} \underbrace{u^2}_{\geqslant 0} \, \mathrm{d}x \geqslant \int_\Omega c \underbrace{u^2}_{\geqslant 0} \, \mathrm{d}x = c \int_\Omega \underbrace{u^2}_{\geqslant 0} \, \mathrm{d}x = c(u,u)_{L^2},$$

and the same property of the non-weighted standard L^2 inner product, which is where ω's boundedness away from zero comes into play. This estimate immediately shows that the standard $W^{1,2}$-norm is weaker than the weighted norm, as it is composed of a sum of L^2 norms.

Essential boundedness of ω yields that the weighted norm in turn is weaker than the standard norm, i. e., we have

$$\sqrt{c}\|u\|_{W^{1,2}} \leqslant \|u\|_{\omega} \leqslant \sqrt{\|\omega\|_{L^\infty}}\|u\|_{W^{1,2}}$$

as one would expect.

You are not expected to turn in your solutions.