Exercise 3

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Homework Problem 3.1. (Norm comparisons)

Suppose that V is a linear space and that $\|\cdot\|_a$ and $\|\cdot\|_b$ are two norms on V such that $\|\cdot\|_a \lesssim \|\cdot\|_b$. Show Lemma 2.12, i. e., the following statements:

(a) For any open ball $B_\varepsilon^{\|\cdot\|_a}(x)$ in the weaker norm $\|\cdot\|_a$, there exists an open ball $B_\delta^{\|\cdot\|_b}(x)$ in the stronger norm $\|\cdot\|_b$ such that $B_\delta^{\|\cdot\|_b}(x) \subseteq B_\varepsilon^{\|\cdot\|_a}(x)$.
(The stronger norm has the smaller/more open balls.)

(b) If $U \subseteq V$ is open in the weaker norm $\|\cdot\|_a$, then U is open in the stronger norm $\|\cdot\|_b$.
(The stronger norm defines the finer topology.)

(c) If $A \subseteq V$ is closed in the weaker norm $\|\cdot\|_a$, then A is closed in the stronger norm $\|\cdot\|_b$.

(d) If $E \subseteq V$ is bounded in the stronger norm $\|\cdot\|_b$, then E is bounded in the weaker norm $\|\cdot\|_a$.

(e) If $K \subseteq V$ is totally bounded in the stronger norm $\|\cdot\|_b$, then K is totally bounded in the weaker norm $\|\cdot\|_a$.

(f) If $K \subseteq V$ is compact in the stronger norm $\|\cdot\|_b$, then K is compact in the weaker norm $\|\cdot\|_a$.

(g) If $\left(x^{(k)}\right)$ converges in the stronger norm $\|\cdot\|_b$, then $\left(x^{(k)}\right)$ converges in the weaker norm $\|\cdot\|_a$ (to the same limit point).

(h) If $\left(x^{(k)}\right)$ is a Cauchy sequence in the stronger norm $\|\cdot\|_b$, then $\left(x^{(k)}\right)$ is a Cauchy sequence in the weaker norm $\|\cdot\|_a$.

Additionally, formulate the corresponding results when $\|\cdot\|_a$ and $\|\cdot\|_b$ are equivalent norms. What kind of result can you expect to hold for completeness?

Homework Problem 3.2. (Non-equivalence of norms on the example of continuous functions)

Let C denote the real function space of restrictions of real valued continuous[GM] functions on $[0, 1] \subseteq \mathbb{R}$ to $(0, 1)$.

(a) Show that the classes defined by the elements of C form a subspace of $L^p(0, 1)$ for all $p \in [1, \infty]$.

(b) Show that $\|\cdot\|_{L^1(0,1)}$ and $\|\cdot\|_{L^\infty(0,1)}$ are not equivalent on C.

(c) Give an example of a subset of C that is compact with respect to $\|\cdot\|_{L^1(0,1)}$ but not with respect to $\|\cdot\|_{L^\infty(0,1)}$.

(d) Show that C is complete with respect to $\|\cdot\|_{L^\infty(0,1)}$ but not complete with respect to $\|\cdot\|_{L^1(0,1)}$.

You are not expected to turn in your solutions.