R. Herzog, G. Müller
Universität Heidelberg

Exercise 3 (Solution)

Date issued: 28th October 2024

Homework Problem 3.1. (Norm comparisons)

Suppose that V is a linear space and that $\|\cdot\|_a$ and $\|\cdot\|_b$ are two norms on V such that $\|\cdot\|_a \lesssim \|\cdot\|_b$. Show Lemma 2.12, i. e., the following statements:

(a) For any open ball $B_\varepsilon^{\|\cdot\|_a}(x)$ in the weaker norm $\|\cdot\|_a$, there exists an open ball $B_\delta^{\|\cdot\|_b}(x)$ in the stronger norm $\|\cdot\|_b$ such that $B_\delta^{\|\cdot\|_b}(x) \subseteq B_\varepsilon^{\|\cdot\|_a}(x)$.

(The stronger norm has the smaller/more open balls.)

(b) If $U \subseteq V$ is open in the weaker norm $\|\cdot\|_a$, then U is open in the stronger norm $\|\cdot\|_b$.

(The stronger norm defines the finer topology.)

(c) If $A \subseteq V$ is closed in the weaker norm $\|\cdot\|_a$, then A is closed in the stronger norm $\|\cdot\|_b$.

(d) If $E \subseteq V$ is bounded in the stronger norm $\|\cdot\|_b$, then E is bounded in the weaker norm $\|\cdot\|_a$.

(e) If $K \subseteq V$ is totally bounded in the stronger norm $\|\cdot\|_b$, then K is totally bounded in the weaker norm $\|\cdot\|_a$.

(f) If $K \subseteq V$ is compact in the stronger norm $\|\cdot\|_b$, then K is compact in the weaker norm $\|\cdot\|_a$.

(g) If $\left(x^{(k)}\right)$ converges in the stronger norm $\|\cdot\|_b$, then $\left(x^{(k)}\right)$ converges in the weaker norm $\|\cdot\|_a$ (to the same limit point).

(h) If $\left(x^{(k)}\right)$ is a Cauchy sequence in the stronger norm $\|\cdot\|_b$, then $\left(x^{(k)}\right)$ is a Cauchy sequence in the weaker norm $\|\cdot\|_a$.

Additionally, formulate the corresponding results when $\|\cdot\|_a$ and $\|\cdot\|_b$ are equivalent norms. What kind of result can you expect to hold for completeness?

Solution.

For the entire proof, assume that $c > 0$ is a constant such that $\|x\|_a \leqslant c\|x\|_b$ for all $x \in V$.

(a) Let $\varepsilon > 0$ be fixed and let $B_\varepsilon^{\|\cdot\|_a}(x)$ be given. Then set $\delta \coloneqq \frac{1}{c}\varepsilon$ and let $y \in B_\delta^{\|\cdot\|_b}(x)$ be given. Then

$$\|y - x\|_a \leqslant c\|y - x\|_b \leqslant c\frac{1}{c}\varepsilon = \varepsilon$$

so $y \in B_\delta^{\|\cdot\|_b}(x) \subseteq B_\varepsilon^{\|\cdot\|_a}(x)$.

(b) Let $U \subseteq V$ be $\|\cdot\|_a$-open. By definition, there is an $\varepsilon > 0$, such that $B_\varepsilon^{\|\cdot\|_a}(x) \subseteq U$. As shown in item (a), the ball $B_\varepsilon^{\|\cdot\|_a}(x)$ contains a ball $B_\delta^{\|\cdot\|_b}(x)$, thus U is also $\|\cdot\|_b$-open.

(c) This is an immediate consequence of item (b) applied to the complement $X \setminus A$ being open in the weaker norm and therefore in the stronger norm.

(d) If there exists a constant $M > 0$ such that $\|x\|_b \leqslant M$ for all $x \in E$, then of course

$$\|x\|_a \leqslant c\|x\|_b \leqslant cM \ \forall x \in E.$$

(e) Let K be totally bounded with respect to $\|\cdot\|_b$ and let $\varepsilon > 0$ be given. By definition, there exists a finite covering of K by the balls $\{B_\delta^{\|\cdot\|_b}(x^{(1)}), \dots, B_\delta^{\|\cdot\|_b}(x^{(n)})\}$ for $\delta \coloneqq \frac{1}{c}\varepsilon$. Since $B_\delta^{\|\cdot\|_b}(x^{(i)}) \subseteq B_\varepsilon^{\|\cdot\|_a}(x^{(1)})$, the set $\{B_\varepsilon^{\|\cdot\|_b}(x^{(1)}), \dots, B_\varepsilon^{\|\cdot\|_a}(x^{(n)})\}$ is also a finite covering of K.

(f) Let K be compact with respect to $\|\cdot\|_b$ and let $(U^{(i)})_{i \in I}$ be a $\|\cdot\|_a$-open covering of K. Then $(U^{(i)})_{i \in I}$ is also a $\|\cdot\|_b$-open covering of K by item (b), so there exists a finite subcover, hence K is also compact in the weaker norm.

(g) Let $x^{(k)} \xrightarrow{b} x$. Then

$$\|x^{(k)} - x\|_a \leqslant c\|x^{(k)} - x\|_b \to 0,$$

i. e., $x^{(k)} \xrightarrow{a} x$.

(h) Let $x^{(k)}$ be b-Cauchy and $\varepsilon > 0$ be arbitrary but fixed. Then for all $\frac{\varepsilon}{c} > 0$, there is a $K_b(\frac{1}{c}\varepsilon)$, such that

$$\|x^{(n)} - x^{(m)}\|_a \leqslant c\|x^{(n)} - x^{(m)}\|_b \le \varepsilon$$

for all $n, m \geqslant K_b(\frac{1}{c}\varepsilon)$.

The statements in non-technical form for equivalent norms are:

(a) For equivalent norms, balls in either norm can be rescaled to lie within the corresponding ball of the other norm.

(b) Equivalent norms induce the same open sets, i. e., to same topology.

(c) Equivalent norms induce the same closed sets, i. e., to same topology.

(d) Equivalent norms induce the same bounded sets.

(e) Equivalent norms induce the same bounded sets.

(f) Equivalent norms induce the same totally bounded sets.

(g) Equivalent norms induce the same compact sets.

(h) Equivalent norms induce the same convergent sequences.

(i) Equivalent norms induce the same Cauchy sequences.

We can only show that for equivalent norms, a set is complete in one norm if and only if it is complete in the other norm. Unintuitively, one can not deduce a one sided implication from items (g) and (h), because there could always be sequences, that are Cauchy in the weaker norm, but not in the stronger norm, see also homework problem 3.2.

Homework Problem 3.2. (Non-equivalence of norms on the example of continuous functions)

Let C denote the real function space of restrictions of real valued continuous[GM] functions on $[0, 1] \subseteq \mathbb{R}$ to $(0, 1)$.

(a) Show that the classes defined by the elements of C form a subspace of $L^p(0, 1)$ for all $p \in [1, \infty]$.

(b) Show that $\|\cdot\|_{L^1(0,1)}$ and $\|\cdot\|_{L^\infty(0,1)}$ are not equivalent on C.

(c) Give an example of a subset of C that is compact with respect to $\|\cdot\|_{L^1(0,1)}$ but not with respect to $\|\cdot\|_{L^\infty(0,1)}$.

(d) Show that C is complete with respect to $\|\cdot\|_{L^\infty(0,1)}$ but not complete with respect to $\|\cdot\|_{L^1(0,1)}$.

Solution.

First off, the definition of C as the set of function restrictions is essentially required because we only consider Lebesgue spaces of functions defined on open sets (because we also want to look at differentiability later).

However, all functions in C correspond to a unique function in $C([0, 1])$ (by continuation of the continuous representative to the boundary).

(a) The linear space structure of C is being induced by the structure of $C([0, 1])$ of course, so we only need to check that C is in fact a subset of $L^\infty(0, 1)$, because Lemma 2.25 states that all essentially bounded functions are p-integrable for all $p \in [1, \infty)$, i. e., it suffices to show that the functions in C are essentially bounded. All functions in C are bounded by the maximum of their unique extension to $[0, 1]$ though, which is also approximated by values on $(0, 1)$, so the $L^\infty(0, 1)$-norm of the restriction coincides with the supremum norm of the extension, i. e., its maximum.

(b) Consider the (restrictions of the) sequence of functions $f^{(k)} \coloneqq x \mapsto x^k$ for $k \in \mathbb{N}$, which lies in C. Its pointwise limit is the (restriction of) $x \mapsto 1, x = 1; 0,$ else, which is discontinuous in 1. Because

$$\|f^{(k)}\|_{L^1(0,1)} = \int_0^1 x^k \, \mathrm{d}x = \frac{1}{k+1} \xrightarrow{k\to\infty} 0 \tag{0.1}$$

but

$$\|f^{(k)}\|_{L^\infty(0,1)} = \sup_{(0,1)} x^k = 1,$$

there can not possibly be a fixed $c > 0$, such that $\|f\|_{L^\infty(0,1)} \leqslant c\|f\|_{L^1(0,1)}$ for all f in C. Accordingly, $\|f\|_{L^\infty(0,1)}$ ist truly stronger on C than $\|f\|_{L^1(0,1)}$ is.

(c) One of these sets is exactly the set of the (classes of the) elements of the sequence constructed in the previous proof, i. e.,

$$K \coloneqq \{[x \mapsto x^k] \mid k \in \mathbb{N}\}.$$

Note that (0.1) shows convergence of the sequence to the zero function.

Any sequence in K either consists of only a finite number of elements of K, thus there is at least one element that is repeated an infinite number of times, leading to a constant (and therefore convergent) subsequence, or it contains an infinite number of elements of the sequence and hence a (potentially reordered) subsequence, which is also $\|\cdot\|_{L^1(0,1)}$-convergent. Thus, the set K is $\|\cdot\|_{L^1(0,1)}$-compact.

The defining sequence itself is of course a sequence in K as well, so showing that this sequence has no $\|\cdot\|_{L^\infty(0,1)}$-convergent subsequence shows that K is not $\|\cdot\|_{L^\infty(0,1)}$-compact. We do that by showing, that any subsequence is not a $\|\cdot\|_{L^1(0,1)}$-Cauchy-sequence. The structure of the argument is: for any given ε, we can look at a point $x_0 \in (0, 1)$ that is close to 1 so that for a given $k \in \mathbb{N}$, the function value of x_0^k is still large so that for $m \geqslant k$ the function value of x_0^m is already small, making the gap large. Formally: Let $\varepsilon \coloneqq \frac{1}{2}$ and let $k \in \mathbb{N}$. Set $x_0 \coloneqq \sqrt[k]{\frac{3}{4}}$ and $n \coloneqq k$. Then for $m \geq \frac{\ln \frac{1}{4}}{\ln x_0}$, we obtain that

$$\|x \mapsto x^n - x^m\|_{L^\infty(0,1)} = \sup_{x\in(0,1)} |x^n - x^m| \geqslant |x_0^n - x_0^m| = \frac{3}{4} - x_0^m \geqslant \frac{3}{4} - \frac{1}{4} = \frac{1}{2} = \varepsilon,$$

which shows that the sequence itself is not a Cauchy sequence but the arguments analogously transfer to a subsequence, showing non-compactness.

(d) Let $f^{(k)}$ be a $L^\infty(0, 1)$-Cauchy-sequence of functions in C. Since these are continuous, this implies that their function values for each $x \in (0, 1)$ are real Cauchy-sequences and therefore converge to values that we can use to define a pointwise limit function $x \mapsto f(x) \coloneqq \lim_{k\to\infty} f^{(k)}(x)$.

This limit function is continuous, as for fixed $\varepsilon > 0$ and $x \in (0,1)$, we have

$$\begin{aligned}|f(x) - f(y)| \leqslant & |f(x) - f^{(k(x))}(x)| + |f^{(k(x))}(x) - f^{(k(x))}(y)| \\ & + |f^{(k(x))}(y) - f^{(k(y))}(y)| + |f^{(k(y))}(y) - f(y)| \\ \leqslant & |f(x) - f^{(k(x))}(x)| + |f^{(k(x))}(x) - f^{(k(x))}(y)| \\ & + \|f^{(k(x))} - f^{(k(y))}\|_{L^\infty(0,1)} + |f^{(k(y))}(y) - f(y)| \\ \leq & 4\frac{\varepsilon}{4},\end{aligned}$$

where (in that order) $k(x)$ is chosen sufficiently large to ensure the first term is small and the third can get small, then y is chosen arbitrarily but sufficiently close to x that the second term is small and finally $k(y)$ is chosen sufficiently large such that the third term and the last term are small.

Now because

$$\begin{aligned}\|f - f^{(k)}\|_{L^\infty(0,1)} &= \sup_{x\in(0,1)} |f(x) - f^{(k)}(x)| \\ &= \sup_{x\in(0,1)} |\lim_{n\to\infty} f^{(n)}(x) - f^{(k)}(x)| \\ &= \lim_{n\to\infty} \sup_{x\in(0,1)} |f^{(n)}(x) - f^{(k)}(x)| \\ &= \lim_{n\to\infty} \|f^{(n)} - f^{(k)}\|_{L^\infty(0,1)} \\ &\leqslant \varepsilon\end{aligned}$$

for k sufficiently large so the last norm becomes small due to the Cauchy sequence property.

Now consider the sequence $f^{(k)}$ of functions in C defined piecewisely by

$$x \mapsto \begin{cases} 0 & x \leqslant \frac{1}{2} - \frac{1}{k} \\ k(x - \frac{1}{2}) & x \in [\frac{1}{2} - \frac{1}{k}, \frac{1}{2}] \\ 1 & x \geqslant \frac{1}{2} \end{cases}.$$

This sequence is an $\|\cdot\|_{L^1(0,1)}$-Cauchy-sequence, because

$$\|f^{(n)} - f^{(m)}\|_{L^1(0,1)} = \int_0^{\frac{1}{m}} mx \,\mathrm{d}x - \int_0^{\frac{1}{n}} nx \,\mathrm{d}x = \frac{1}{2m} - \frac{1}{2n} = \frac{n-m}{2nm} \leqslant \frac{2n}{2n^2} = \frac{1}{n} \xrightarrow{n\to\infty} 0.$$

This sequence obviously converges to $\chi_{[\frac{1}{2},1]} \in L^1((0,1))$ (the indicator function of the set $[\frac{1}{2},1)$) in a pointwise sense, and, because

$$\|f^{(k)}\|_{L^1(0,1)} = \frac{1}{2k} \xrightarrow{k\to 0} 0,$$

it also converges to the indicator function in the L_1 sense, but $\chi_{[\frac{1}{2},1]}$ is not continuous, so there is no limit point in C for this sequence.

You are not expected to turn in your solutions.