R. Herzog, G. Müller
Universität Heidelberg

Exercise 2

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Homework Problem 2.1. (Convergent and Cauchy Sequences)

Suppose that $(V, \|\cdot\|)$ is a normed linear space and that $\big(x^{(k)}\big)$ is a sequence in V. Show Lemma 2.6, i. e., the following statements.

(a) Suppose that $\big(x^{(k)}\big)$ converges. Then its limit is unique.

(b) Suppose that $\big(x^{(k)}\big)$ converges. Then it is a Cauchy sequence.

Homework Problem 2.2. (Completeness of Banach space subsets)

Let $(V, \|\cdot\|)$ be a Banach space and let $A \subseteq V$. Show that A is complete if and only if A is closed.

Homework Problem 2.3. (Space Completion via Cauchy Sequences)

(a) Explain why $x^{(k)} \coloneqq \big(1 + \frac{1}{k}\big)^k$ is an example that shows incompleteness of $(\mathbb{Q}, |\cdot|)^{\text{GM}}$. **Hint:** Assume standard analysis knowledge here, i. e., that this sequence converges to $e \in \mathbb{R} \setminus \mathbb{Q}$ in the real numbers with respect to the absolute value.

(b) Suppose that $(V, \|\cdot\|)$ is a normed real$^{\text{GM}}$ linear space and consider the quotient space

$$\widetilde{V} \coloneqq \Big\{(x^{(k)}) \,\Big|\, (x^{(k)}) \text{ is a } V\text{-Cauchy sequence}\Big\} \,/\, \Big\{(y^{(k)}) \,\Big|\, (y^{(k)}) \text{ is a } V\text{-null-sequence}\Big\}$$

whose elements are the cosets $[(x^{(k)})]$ for V-Cauchy-sequences $(x^{(k)})$ of the form

$$[(x^{(k)})] = \{(x^{(k)}) + (y^{(k)}) \mid (y^{(k)}) \text{ is a } V\text{-null-sequence}\}.$$

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(i) Show that $(\widetilde{V}, \|\cdot\|_{\widetilde{V}})$ with

$$\left\|\left[(x^{(k)})\right]\right\|_{\widetilde{V}} \coloneqq \lim_{k\to\infty} \|x^{(k)}\|_V$$

is a normed space.

(ii) Show that $(\widetilde{V}, \|\cdot\|_{\widetilde{V}})$ is complete. **Hint:** Consider a diagonal sequence.

(iii) Show that the mapping

$$E\colon V \ni x \mapsto [(x, x, x, \dots)] \in \widetilde{V}$$

is an isometric embedding of $(V, \|\cdot\|_V)$ into $(\widetilde{V}, \|\cdot\|_{\widetilde{V}})$, where $E(V)$ is dense in $\widetilde{V}$.

(c) Suppose that $(V, \|\cdot\|_V)$ is a normed linear space that is densely and isometrically embedded into a complete space $(\widetilde{V}, \|\cdot\|)_{\widetilde{V}}$. Show that $(\widetilde{V}, \|\cdot\|)_{\widetilde{V}}$ is unique up to isometric isomorphy.

You are not expected to turn in your solutions.