Exercise 2 (Solution)

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Homework Problem 2.1. (Convergent and Cauchy Sequences)

Suppose that $(V, \|\cdot\|)$ is a normed linear space and that $\big(x^{(k)}\big)$ is a sequence in V. Show Lemma 2.6, i. e., the following statements.

(a) Suppose that $\big(x^{(k)}\big)$ converges. Then its limit is unique.

(b) Suppose that $\big(x^{(k)}\big)$ converges. Then it is a Cauchy sequence.

Solution.

Both subitems are essentially standard analysis results.

(a) Assume that $(x^{(k)})$ is convergent to x and to $\tilde{x}$ in V. Then for any $\varepsilon > 0$, there is $k_0 \in \mathbb{N}$ such that

$$\|x^{(k_0)} - x\| \le \varepsilon \qquad \text{and} \qquad \|x^{(k_0)} - \tilde{x}\| \le \varepsilon.$$

Accordingly

$$\|x - \tilde{x}\| \le \|x - x^{(k_0)}\| + \|x^{(k_0)} - \tilde{x}\| \le 2\varepsilon$$

and since ε can be chosen arbitrarily, we obtain $\|x - \tilde{x}\| = 0$ and therefore $x = \tilde{x}$, i. e., uniqueness of the limit point.

(b) Let $x \coloneqq \lim_{k\to\infty} x^{(k)}$ and let $\varepsilon > 0$ be given arbitrarily. By definition, there exists a $k_0 \in \mathbb{N}$ such that

$$\|x^{(k)} - x\| \le \varepsilon \; \forall k \geqslant k_0.$$

Accordingly, for any $n, m \geqslant k_0$, we have that

$$\|x^{(n)} - x^{(m)}\| \le \|x^{(n)} - x\| + \|x - x^{(m)}\| = 2\varepsilon.$$

Homework Problem 2.2. (Completeness of Banach space subsets)

Let $(V, \|\cdot\|)$ be a Banach space and let $A \subseteq V$. Show that A is complete if and only if A is closed.

Solution.

"$\Leftarrow$": Let $(x^{(k)})$ be a V sequence that lies in A. Since V is a complete space, $(x^{(k)})$ converges to an $x \in V$. Since A is closed, $x \in A$ so A is complete.

"$\Rightarrow$": Let $(x^{(k)})$ be a convergent subsequence of A. Since $(x^{(k)})$ is convergent, it is a Cauchy sequence and because A is complete, $(x^{(k)})$ converges to an $x \in A$ and the limit point is unique, therefore A is closed.

Homework Problem 2.3. (Space Completion via Cauchy Sequences)

(a) Explain why $x^{(k)} \coloneqq \left(1 + \frac{1}{k}\right)^k$ is an example that shows incompleteness of $(\mathbb{Q}, |\cdot|)^{\text{GM}}$. **Hint:** Assume standard analysis knowledge here, i. e., that this sequence converges to $e \in \mathbb{R} \setminus \mathbb{Q}$ in the real numbers with respect to the absolute value.

(b) Suppose that $(V, \|\cdot\|)$ is a normed real$^{\text{GM}}$ linear space and consider the quotient space

$$\widetilde{V} \coloneqq \left\{(x^{(k)}) \,\middle|\, (x^{(k)}) \text{ is a } V\text{-Cauchy sequence}\right\} \,/\, \left\{(y^{(k)}) \,\middle|\, (y^{(k)}) \text{ is a } V\text{-null-sequence}\right\}$$

whose elements are the cosets $[(x^{(k)})]$ for V-Cauchy-sequences $(x^{(k)})$ of the form

$$[(x^{(k)})] = \{(x^{(k)}) + (y^{(k)}) \mid (y^{(k)}) \text{ is a } V\text{-null-sequence}\}.$$

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(*i*) Show that $(\widetilde{V}, \|\cdot\|_{\widetilde{V}})$ with

$$\left\|[(x^{(k)})]\right\|_{\widetilde{V}} \coloneqq \lim_{k\to\infty} \|x^{(k)}\|_V$$

is a normed space.

(*ii*) Show that $(\widetilde{V}, \|\cdot\|_{\widetilde{V}})$ is complete. **Hint:** Consider a diagonal sequence.

(*iii*) Show that the mapping

$$E\colon V \ni x \mapsto [(x, x, x, \dots)] \in \widetilde{V}$$

is an isometric embedding of $(V, \|\cdot\|_V)$ into $(\widetilde{V}, \|\cdot\|_{\widetilde{V}})$, where $E(V)$ is dense in $\widetilde{V}$.

(c) Suppose that $(V, \|\cdot\|_V)$ is a normed linear space that is densely and isometrically embedded into a complete space $(\widetilde{V}, \|\cdot\|)_{\widetilde{V}}$. Show that $(\widetilde{V}, \|\cdot\|)_{\widetilde{V}}$ is unique up to isometric isomorphy.

Solution.

(a) The elements of the sequence are all finite products of obviously rational numbers and therefore rational. The sequence is well-known to converge to e, which a real, irrational number, i. e., in $\mathbb{R} \setminus \mathbb{Q}$. Accordingly, as shown in homework problem 2.1, the sequence is convergent and therefore Cauchy in $\mathbb{R}$ with elements in $\mathbb{Q}$ and with respect to the absolute value. It of course remains Cauchy in $\mathbb{Q}$ with respect to the absolute value, but it is not convergent because its $\mathbb{R}$-limit is not in $\mathbb{Q}$, meaning $\mathbb{Q}$ is not complete.

(b) The elements of $\widetilde{V}$ are cosets of Cauchy sequences that have the form of the Minkowski sums $(x^{(k)}) + \left\{(y^{(k)}) \,\middle|\, (y^{(k)}) \text{ is a } V\text{-null-sequence}\right\}$.

To show that this is in fact a linear space (with the quotient operations), it suffices to show that Cauchy and nullsequences are both subspaces of the sequences on V because nullsequences are obviously convergent and therefore Cauchy. Both facts are pretty standard facts from Analysis I that we won't repeat here.

(i) First off, note that the norm is in fact well-defined, as the norm-sequence of a Cauchy sequence converges in the reals, because of the inverse triangle inequality for $\|\cdot\|_V$, we have that

$$\left|\left\|x^{(k)}\right\|_V - \left\|x^{(l)}\right\|_V\right| \le \left\|x^{(k)} - x^{(l)}\right\|_V$$

for any V-sequence $x^{(k)}$, where the right hand sides form a real nullsequence, showing that $\left\|x^{(k)}\right\|_V$ form a real Cauchy Sequence, which converges because $\mathbb{R}$ is complete.

Now for another sequence $(y^{(k)}) \in \left[(x^{(k)})\right]$, we have that

$$\left|\left\|x^{(k)}\right\|_V - \left\|y^{(k)}\right\|_V\right| \le \left\|x^{(k)} - y^{(k)}\right\|_V \xrightarrow{k\to\infty} 0,$$

so the definition is independent of the representative of the classes.

It remains to show that $\|\cdot\|_{\widetilde{V}}$ is in fact a norm on $\widetilde{V}$.

The zero-class is exactly the space of nullsequences, whose limit in norm is zero as well, essentially by definition. It yields nonnegative numbers as the limit of nonnegative number remains nonnegative. Now let $\left[(x^{(k)})\right]$ be a class of sequences with $\left\|[x^{(k)}]\right\|_{\widetilde{V}} = 0$, then $\lim_{k\to\infty} \left\|x^{(k)}\right\|_V = \left\|\left[(x^{(k)})\right]\right\|_{\widetilde{V}} = 0$, meaning that $x^{(k)}$ is a nullsequence and therefore $\left[(x^{(k)})\right]$ is the zero-class in $\widetilde{V}$. This shows positive definiteness.

For positive homogeneity, we observe that, for $\alpha \in \mathbb{R}$, we have

$$\begin{aligned}\left\|\alpha\left[(x^{(k)})\right]\right\|_{\widetilde{V}} &= \left\|\left[\alpha(x^{(k)})\right]\right\|_{\widetilde{V}} = \lim_{k\to\infty}\left\|\alpha(x^{(k)})\right\|_V = \lim_{k\to\infty}|\alpha|\left\|(x^{(k)})\right\|_V \\ &= |\alpha| \lim_{k\to\infty}\left\|(x^{(k)})\right\|_V = |\alpha|\left\|[(x^{(k)})]\right\|_{\widetilde{V}}.\end{aligned}$$

Similarly, subadditivity follows, for V-Cauchy sequences $(x^{(k)})$ and $(y^{(k)})$, from the subadditivity of $\|\cdot\|_V$ and monotonicity of the limit process, i. e., from

$$\begin{aligned}\left\|(x^{(k)}) + (y^{(k)})\right\|_{\widetilde{V}} &= \lim_{k\to\infty} \|x^{(k)} + y^{(k)}\|_V \le \lim_{k\to\infty} \|x^{(k)}\|_V + \|y^{(k)}\|_V = \lim_{k\to\infty} \|x^{(k)}\|_V + \lim_{k\to\infty} \|y^{(k)}\|_V \\ &= \left\|(x^{(k)})\right\|_{\widetilde{V}} + \left\|(y^{(k)})\right\|_{\widetilde{V}}.\end{aligned}$$

(ii) Let $\left[(x^{(k)})\right]^{(K)}$ be a $\widetilde{V}$-Cauchy-sequence (of Cauchy sequence classes).
Since all the (fixed) representatives $\left(x^{(k)}\right)$ are Cauchy sequences themselves, we have that for every $K \in \mathbb{N}$, there is a $k_K \in \mathbb{N}$, such that $\left\|x^{(n)\,(K)} - x^{(m)\,(K)}\right\|_V \leqslant \frac{1}{K}$ for all $n, m \geqslant k_K$.
We fix the sequence $(y^{(K)}) \coloneqq \left(x^{(k_K)\,(K)}\right)_K$ and will now show that $[(y^{(K)})] \in \widetilde{V}$ is the limit of $\left[(x^{(k)})\right]^{(K)}$ in $\widetilde{V}$. To that end, let $\varepsilon > 0$.
Because $\left[(x^{(k)})\right]^{(K)}$ be a $\widetilde{V}$ is a Cauchy sequence, there is a K_0, such that

$$\left\|\left[\left(x^{(k)}\right)\right]^{(N)} - \left[\left(x^{(k)}\right)\right]^{(M)}\right\|_{\widetilde{V}} \le \varepsilon \quad \text{for all } N, M \ge K_0.$$

Because of the limiting definition of the $\widetilde{V}$ norm, this means there for any $N, M \ge K_0$, there is $k_0(N, M)$ such that

$$\|x^{(k)\,(N)} - x^{(k)\,(M)}\|_V \leqslant 2\varepsilon$$

for all $k \geqslant k_0(N, M)$.
We can therefore estimate

$$\left\|y^{(N)} - y^{(M)}\right\|_V \le \left\|y^{(N)} - x^{(k)\,(N)}\right\|_V + \left\|x^{(k)\,(N)} - x^{(k)\,(M)}\right\|_V + \left\|x^{(k)\,(M)} - y^{(M)}\right\|_V \leqslant 4\varepsilon$$

for all $N, M \geqslant K_0$ and $k = \max(k_0(N, M), k_N, k_M)$, therefore $\left(y^{(k)}\right)$ is a Cauchy sequence.
Now we only need to show that $\left[(x^{(k)})\right]^{(K)}$ in fact converges to $\left[(y^{(K)})\right]$. Since $(y^{(K)})$ is a Cauchy sequence, we know that

$$\left\|y^{(N)} - y^{(M)}\right\|_V \leqslant \varepsilon$$

for all $N, M \geqslant K_1(\varepsilon)$ for a $K_1(\varepsilon)$. Additionally, by definition of k_K,

$$\left\|x^{(k)\,(K)} - y^{(K)}\right\|_V = \left\|x^{(k)\,(K)} - x^{(k_K)\,(K)}\right\|_V \leqslant \frac{1}{K}$$

for all $k \geqslant k_K$. Accordingly,

$$\left\|\left(x^{(k)}\right)^{(K)} - y^{(k)}\right\|_V \leqslant \left\|\left(x^{(k)}\right)^{(K)} - y^{(K)}\right\|_V + \left\|y^{(K)} - y^{(k)}\right\|_V \leq 2\varepsilon$$

for $K \geqslant \max(K_1(\varepsilon), \frac{1}{\varepsilon})$, $k \geqslant \max(K_1(\varepsilon), k_K)$.

Hence, by the definition of $\|\cdot\|_{\widetilde{V}}$, we have that

$$\left\|\left[\left(x^{(k)}\right)\right]^{(K)} - \left[\left(y^{(k)}\right)\right]\right\|_{\widetilde{V}} = \lim_{k\to\infty} \left\|\left(x^{(k)}\right)^{(K)} - y^{(k)}\right\|_V \leqslant 2\varepsilon$$

for all $K \geqslant \max(K_1(\varepsilon), \frac{1}{\varepsilon})$.

(*iii*) Linearity of the mapping as well as the mapping properties and its well-definedness are clear. We denote the embedding by $E\colon V \to \widetilde{V}$. Then, isometry follows from

$$\|E(x)\|_{\widetilde{V}} = \lim_{k\to\infty} \|x\|_V = \|x\|_V.$$

Density is by construction, as an element of $\widetilde{V}$, say, $\left[(x^{(k)})\right]$, can be approximated by the sequence of classes of constant sequences of the elements $x^{(k)}$ of the representative.

(c) Assume that $(V, \|\cdot\|_V)$ is densely and isometrically embedded into $(\widetilde{V_1}, \|\cdot\|_{\widetilde{V_1}})$ and $(\widetilde{V_2}, \|\cdot\|_{\widetilde{V_2}})$ via linear isometries $E_1\colon V \to \widetilde{V_1}$ and $E_2\colon V \to \widetilde{V_2}$, respectively.

Since E_1 and E_2 are isometries, their kernels are trivial, hence they are invertible. Therefore, $f \coloneqq E_2 \circ E_1^{-1}\colon E_1(V) \to E_2(V)$ is an isometric isomorphism. Since $E_1(V)$ is dense in $\widetilde{V_1}$, we can extend f to $F\colon \widetilde{V_1} \to \widetilde{V_2}$ as follows:

Let $v_1 \in \widetilde{V_1}$ and let $(x^{(k)})$ be a V-sequence, such that $E_1(x^{(k)}) \xrightarrow{k\to\infty} v_1$ in $\widetilde{V_1}$. Then $E_1(x^{(k)})$ is also a Cauchy-sequence and because of isometry, $f(x^{(k)}) = E_2(x^{(k)})$ is also a Cauchy-sequence, that converges (because of completeness of $\widetilde{V_2}$) to a $v_2 \in \widetilde{V_2}$. For any other V-sequence $(y^{(k)})$ with $E_1(y^{(k)}) \xrightarrow{k\to\infty} v_1$, we also get convergence of $f(E_1(y^{(k)}))$ to a z_2 in $\widetilde{V_2}$ and

$$\begin{aligned}\|z_2 - v_2\|_{\widetilde{V_2}} &= \|\lim_{k\to\infty} E_2(y^{(k)}) - \lim_{k\to\infty} E_2(x^{(k)})\|_{\widetilde{V_2}} = \lim_{k\to\infty} \|E_2(y^{(k)}) - E_2(x^{(k)})\|_{\widetilde{V_2}} \\ &= \lim_{k\to\infty} \|E_1(y^{(k)}) - E_1(x^{(k)})\|_{\widetilde{V_1}} = \|\lim_{k\to\infty} E_1(y^{(k)}) - \lim_{k\to\infty} E_1(x^{(k)})\|_{\widetilde{V_1}} = \|v_1 - v_1\|_{\widetilde{V_1}} = 0,\end{aligned} \tag{0.1}$$

so the limit point in $\widetilde{V_2}$ is independent of the approximation Cauchy sequence, due to isometry. We can therefore define the extension of $f\colon E_1(V) \to E_2(V)$ accordingly as

$$F\colon \widetilde{V_1} \to \widetilde{V_2}, \quad F(v_1) = v_2.$$

Linearity of F is inherent in the construction and the arguments in (0.1) can be immediately extended to yield the isometry property of F, when letting one of the arguments equal 0.

Isometry implies injectivity of F, we are only missing surjectivity of F. Let $v_2 \in \widetilde{V}_2$. With the same arguments as before (the situation is symmetric), we can obtain a V-sequence $(x^{(k)})$ such that $E_2(x^{(k)}) \to v_2$ and $E_1(x^{(k)}) \to v_1$ for a v_1 in $\widetilde{V}_1$. Then the constuction of F immediately yields

$$F(v_1) = v_2,$$

i. e., surjectivity of F.

You are not expected to turn in your solutions.