Exercise 1 (Solution)

Date issued: 14th October 2024

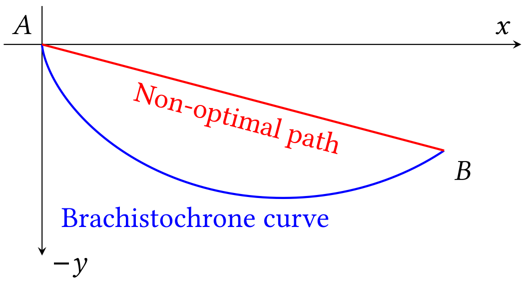
Homework Problem 1. (The brachistochrone problem)

The brachistochrone problem (Example 1.1) is the problem of finding the time-optimal trajectory of a mass moving from point A to point B under gravitational load.

Derive the corresponding optimization problem (1.1), i. e., expand the description from the lecture notes.

Solution.

We fix $A = (0, 0)$ and $B = (b_1, b_2)$ with $b_1 > 0$, $b_2 \leqslant 0$. The gravitational constant is denoted by $g > 0$. The mass is $m > 0$ and the mass is at rest at A at the initial time 0.



For any time interval $[0, T]$ with $T > 0$, we consider the mass to be travelling on the trajectory

$$z(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} x(t) \\ \gamma(x(t)) \end{pmatrix}.$$

Note that this is the first direct integration of a physical side constraint into the problem formulation, because we fix the y-component of the trajectory to the prediscribed γ, whose optimal form we will be searching for. Aside from this physical constraint, the second constraint is energy conservation. Since the mass is at rest at the initial time, both its kinetic and potential energy are 0. We therefore obtain the equation

$$E_{\text{kin}}(t) + E_{\text{pot}}(t) = 0$$

and substituting $E_{\text{kin}}(t) = \frac{1}{2}mv(t)^2$ as well as $E_{\text{pot}} = mgy(t)$ for the speed $v(t)$, we obtain

$$0 = E_{\text{kin}}(t) + E_{\text{pot}}(t) = \frac{1}{2}mv(t)^2 + mgy(t) = \frac{1}{2}m\,(\dot{x}(t))^2 \left(1 + \left(\frac{d}{dx}\gamma(x(t))\right)^2\right) + mg\gamma(x(t))$$

For an appropriate function space Z to choose the trajectories from, the optimization problem therefore is

$$\begin{aligned} \text{Minimize} \quad & T, \quad \text{where } z \in Z \\ \text{s. t.} \quad & z(0) = A \\ \text{and} \quad & z(T) = B \\ \text{and} \quad & 0 = \frac{1}{2}mv(t)^2 + mgy(t) \quad \forall t \in [0, T]. \end{aligned} \tag{0.1}$$

Assuming sufficient regularity and strict monotonicity of the trajectory component x, we can reparameterize the problem with respect to x, i. e., apply an integral transformation from $[0, T]$ with respect to t to $[0, b_1]$ with respect to x, i. e.,

$$T = \int_0^T 1\,\mathrm{d}t = \int_{x^{-1}(0)}^{x^{-1}(b_1)} \frac{\dot{x}(t)}{\dot{x}(t)}\,\mathrm{d}t = \int_{0=x(0)}^{b_1=x(T)} \frac{1}{\dot{x}(t)}\,\mathrm{d}x$$

where we now only need a representation of $\dot{x}(t)$ dependent on x, which is supplied by the energy conservation constraint. The mass travelling on the trajectory $z(t)$ has the velocity

$$\begin{aligned} v &:= \|\dot{z}(t)\|_2 = \sqrt{\dot{x}(t)^2 + \dot{y}(t)^2} = \sqrt{\dot{x}(t)^2 + \left(\frac{d}{dt}\gamma(x(t))\right)^2} \\ &= \sqrt{\dot{x}(t)^2 + \left(\frac{d}{dx}\gamma(x(t))\dot{x}(t)\right)^2} = |\dot{x}(t)|\sqrt{1 + \left(\frac{d}{dx}\gamma(x(t))\right)^2}. \end{aligned}$$

and therefore

$$\dot{x}(t) = \frac{\sqrt{-2g\gamma(x(t))}}{\sqrt{1 + \left(\frac{d}{dx}\gamma(x(t))\right)^2}},$$

which means

$$\frac{1}{\dot{x}(t)} = \frac{\sqrt{1 + \left(\frac{d}{dx}y(x(t))\right)^2}}{\sqrt{-2gy(x(t))}}.$$

Expanding the side constraints on z yields the optimization problem

$$\begin{aligned} \text{Minimize} \quad & \int_0^{b_1} \frac{\sqrt{1 + y'(x)^2}}{\sqrt{-2\,g\,y(x)}}\,\mathrm{d}x, \quad \text{where } y \in X \\ \text{s.t.} \quad & y(0) = 0 \\ \text{and} \quad & y(a) = b \\ \text{as well as} \quad & y \leqslant 0 \text{ on } [0, b_1] \end{aligned} \tag{1.1}$$

for an appropriate function space X. The energy constraint is directly incorporated due to the utilized representation of the velocity.

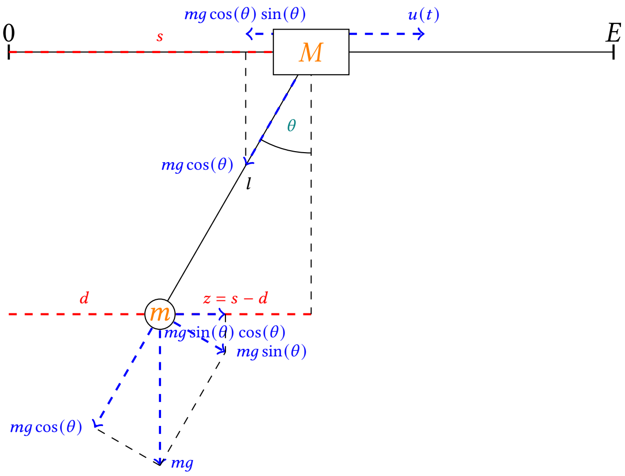
Homework Problem 2. (The crane-trolley-problem)

The crane-trolley-problem (Example 1.5) is the problem of finding the time-optimal control for steering the state-trajectory of a system comprised of a mass connected to a vertically fixed carriage system (which control forces can act on horizontally) by a massless fixed rod into a target state under gravitational load.

Derive the corresponding optimization problem (1.9), i. e., expand the description from the lecture notes.

Solution.

The modelling part of the crane-trolley-problem is pretty basic mechanics. Below you can find a sketch of the situation with labels of all relevant quantities.



where

(a) M is the mass of the carriage, m is the mass of the hanging load
(b) s is the x-displacement of the carriage, d that of the load and z its relative x-displacement
(c) l is the length of the massless rod connecting the carriage and the load, E is the final x-position that both the carriage and the load are supposed to end up at in rest
(d) $u(t)$ are the forces applied to the carriage by us at time t, gravitational forces act on the load with gravitational constant g
(e) Θ is the angle of the rod at the carriage relative to vertical

We need to compute the combined forces acting at the carriage and the load, respectively, to obtain their acceleration. The carriage being vertically fixed means that we can reduce the system to the horizontal dynamics.

We have three external forces acting on the system. The first forces are the gravitational forces on the carriage, which is assumed to be fixed vertically, so the fixing system compensates those forces so they don't have any effect on the system – we won't be addressing them further. The second force is that of the control, acting horizontally on the carriage. The last one is the gravitational force acting on the load. We are only interestend in horizontal force components. The massless rod will transfer some of the forces on either of its ends to the other end.

Starting at the top, we have the forces $u(t)$ acting on the carriage in horizontal direction. Since the rod has fixed length, it will transfer the partial forces of $u(t)$ acting parallely to its orientation (i. e. $u(t)\sin(\theta)$) down to the load, where the horizontal part contributes to the forces at the load with magnitude $u(t)\sin^2(\theta)$. At the bottom, the gravitational force is acting on the load, i. e. $-mge_y$ where the y coordinate corresponds to the vertical one. The components acting parallely to the rod will be transferred to additionally act on the carriage ($mg\sin(\theta)\cos(\theta)$) while the part acting normal to the rod (i. e. tangential to the circle around the carriage that the load can move on) will accelerate the load, where the horizontal component has the same magnitude as the forces transmitted.

Accordingly, the linearized system of ordinary differential equations describing the horizontal dynamics is

$$\begin{aligned} M\ddot{s} &= -mg\cos(\theta)\sin(\theta) + u(t) \\ m\ddot{d} &= mg\sin(\theta)\cos(\theta) + u(t)\sin^2(\theta) \end{aligned}$$

Now we know that $\sin(\theta) = \frac{s-d}{l} = \frac{z}{l} \approx \theta$ and we assume small angles θ and drop all terms of at least second order in θ (linearizing cos/sin at $\theta = 0$). This removes the $\sin^2$ term and yields $\cos(\theta) \approx 1$ to yield the final system

$$\begin{aligned} \ddot{s} &= -\frac{m}{M}\frac{g}{l}z + \frac{1}{M}u(t) \\ \ddot{z} = \ddot{s} - \ddot{d} &= -\frac{(m+M)}{M}\frac{g}{l}z + \frac{1}{M}u(t). \end{aligned}$$

Reformulating the system of second order ODEs as a system of first order and adding the target functional, we obtain precisely the system

$$\begin{aligned} \text{Minimize} \quad & \int_0^T 1\,\mathrm{d}t, \quad \text{where } (u, x, T) \in U \times X \times \mathbb{R} \\ \text{s. t.} \quad & \dot{x} = A\,x + B\,u \quad \text{in } [0, T] \\ \text{and} \quad & x(0) = (0, 0, 0, 0)^\mathsf{T} \\ \text{and} \quad & x(T) = (E, 0, 0, 0)^\mathsf{T} \\ \text{as well as} \quad & T > 0. \end{aligned} \tag{1.8}$$

We can renormalize the unknown time interval $[0, T]$ to the fixed interval $[0, 1]$. Replacing the unknowns x and u by their counterparts on the fixed interval, the dynamics need to be rescaled and the problem becomes

$$\begin{aligned} \text{Minimize} \quad & \int_0^1 T\,\mathrm{d}t, \quad \text{where } (u, x, T) \in U \times X \times \mathbb{R} \\ \text{s. t.} \quad & \dot{x} = \frac{1}{T}(A\,x + B\,u) \quad \text{in } [0, 1] \\ \text{and} \quad & x(0) = (0, 0, 0, 0)^\mathsf{T} \\ \text{and} \quad & x(1) = (E, 0, 0, 0)^\mathsf{T} \\ \text{as well as} \quad & T > 0. \end{aligned} \tag{1.9}$$

You are not expected to turn in your solutions.