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Exercise 13

Date issued: 8th July 2024
Date due: 16th July 2024

Homework Problem 13.1 (Detecting convergence in primal-dual active set strategies) 6 Points

Consider the primal-dual active set strategy (semismooth Newton, Algorithm 11.10) for the lower bound constrained QP from the lecture notes with the iterates $(d^{(k)}, \mu^{(k)}, \lambda^{(k)})$, initialized with some $(d^{(0)}, \mu^{(0)}, \lambda^{(0)})$.

(i) Show that the residual $F(d^{(k)}, \mu^{(k)}, \lambda^{(k)})$ is nonzero only in its second component for $k \geq 1$.

(ii) Prove that when

$$\mathcal{A}(d^{(k)}, \mu^{(k)}) = \mathcal{A}(d^{(k+1)}, \mu^{(k+1)})$$

for some $k \in \mathbb{N}$ (the primal-dual active index sets coincide for two consecutive iterations) then $(d^{(k+1)}, \mu^{(k+1)}, \lambda^{(k+1)})$ is a solution of the constrained QP.

Homework Problem 13.2 (Differentiability of the ℓ_1-merit function) 2 Points

Verify that the directional derivative

$$\pi_1'(x; d) \coloneqq \lim_{t \searrow 0} \frac{\pi_1(x + t\, d) - \pi_1(x)}{t}$$

of the ℓ_1-penalty part

$$\pi_1(x) \colon \mathbb{R}^n \to \mathbb{R}, \quad \pi_1(x) \coloneqq \sum_{i=1}^{n_{\text{ineq}}} \max\{0, g_i(x)\} + \sum_{j=1}^{n_{\text{eq}}} |h_i(x)|$$

of the ℓ_1-merit function exists everywhere and is given by

$$\begin{aligned} \pi_1'(x;d) = &\sum_{\substack{i=1\\ g_i(x)<0}}^{n_{\text{ineq}}} 0 \quad + \sum_{\substack{i=1\\ g_i(x)=0}}^{n_{\text{ineq}}} \max\{0, g_i'(x)\,d\} \quad + \sum_{\substack{i=1\\ g_i(x)>0}}^{n_{\text{ineq}}} g_i'(x)\,d \\ &+ \sum_{\substack{j=1\\ h_j(x)<0}}^{n_{\text{eq}}} -h_j'(x)\,d \quad + \sum_{\substack{j=1\\ h_j(x)=0}}^{n_{\text{eq}}} |h_j'(x)\,d| \quad + \sum_{\substack{j=1\\ h_j(x)>0}}^{n_{\text{eq}}} h_j'(x)\,d \end{aligned} \tag{12.2}$$

for $d \in \mathbb{R}^n$.

Homework Problem 13.3 (Penalty reformulation of infeasible SQP-subproblems) 1 Points

Show that the penalty reformulation

$$\begin{aligned} \text{Minimize} \quad & \frac{1}{2} d^\top A\, d - b^\top d + \gamma \left[\mathbf{1}^\top v + \mathbf{1}^\top w + \mathbf{1}^\top t\right], \quad \text{where } (d, v, w, t) \in \mathbb{R}^n \times \mathbb{R}^{n_{\text{eq}}} \times \mathbb{R}^{n_{\text{eq}}} \times \mathbb{R}^{n_{\text{ineq}}} \\ \text{subject to} \quad & B_{\text{eq}}\, d - c_{\text{eq}} = v - w \\ \text{and} \quad & B_{\text{ineq}}\, d - c_{\text{ineq}} \le t \\ \text{as well as} \quad & v \ge 0, \; w \ge 0, \; t \ge 0 \end{aligned} \tag{12.10}$$

of the SQP-subproblem-type problem

$$\begin{aligned} \text{Minimize} \quad & \frac{1}{2} d^\top A\, d - b^\top d, \quad \text{where } d \in \mathbb{R}^n \\ \text{subject to} \quad & B_{\text{eq}}\, d - c_{\text{eq}} = 0 \\ \text{and} \quad & B_{\text{ineq}}\, d - c_{\text{ineq}} \le 0 \end{aligned} \tag{12.9}$$

is always feasible.

Homework Problem 13.4 (Smoothness properties of exact penalty functions) 6 Points

Consider the constrained optimization problem

$$\text{minimize} \quad f(x) \quad \text{where} \quad x \in \mathcal{F} \tag{P}$$

for a functional $f\colon \mathbb{R}^n \to \mathbb{R}$ and a nonempty feasible set $\mathcal{F} \subseteq \mathbb{R}^n$. Further, define the penalized (unconstrained) problems

$$\text{minimize} \quad f(x) + \gamma\pi(x) \quad \text{where} \quad x \in \mathbb{R}^n \tag{P$_\gamma$}$$

for a penalty function $\pi\colon \mathbb{R}^n \longrightarrow \mathbb{R}$ and a (penalty) parameter $\gamma > 0$.

Note: A penalty function is defined as satisfying $\pi(x) = 0$ for $x \in \mathcal{F}$ and $\pi(x) > 0$ for $x \in \mathbb{R}^n \setminus \mathcal{F}$.

Show the following:

(i) If $x^* \in \mathcal{F}$ is a local/global solution for (P_γ) for a $\gamma^* > 0$, then it is a local/global solution for (P) and for (P_γ) for any $\gamma \geq \gamma^*$.

(ii) If there exist a $\gamma^* > 0$ and an $x^* \in \mathbb{R}^n$, such that x^* is a global solution of (P_γ) for all $\gamma \geq \gamma^*$, then x^* is a global solution to (P).

(iii) Let f be differentiable. If $x^* \in \mathcal{F}$ is a local solution to (P) and to (P_γ) for a $\gamma^* > 0$, then π is not differentiable at x^* or $f'(x^*) = 0$.

What does Statement (iii) mean for exact penalization methods in general?

Please submit your solutions as a single pdf and an archive of programs via moodle.