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Exercise 12

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Homework Problem 12.1 (CQs are invariant under Slack Transformation) 10 Points

We can reformulate the original nonlinear problem

$$\left.\begin{array}{rll} \text{Minimize} & f(x) & \text{where } x \in \mathbb{R}^n \\ \text{subject to} & g_i(x) \le 0 & \text{for } i = 1, \ldots, n_{\text{ineq}} \\ \text{and} & h_j(x) = 0 & \text{for } j = 1, \ldots, n_{\text{eq}} \end{array}\right\} \tag{5.1}$$

by introducing a so called **slack variable** $s \in \mathbb{R}^{n_{\text{ineq}}}$ to obtain the simple one-sided box-constrained problem

$$\left.\begin{array}{rll} \text{Minimize} & f(x) & \text{where } (x, s) \in \mathbb{R}^{n \times n_{\text{ineq}}} \\ \text{subject to} & g_i(x) + s = 0 & \text{for } i = 1, \ldots, n_{\text{ineq}} \\ \text{and} & -s \le 0 & \\ \text{and} & h_j(x) = 0 & \text{for } j = 1, \ldots, n_{\text{eq}} \end{array}\right\}. \tag{5.1$_\text{s}$}$$

(*i*) Derive the KKT-system of (5.1$_\text{s}$) and show that there is a one-to-one connection between the solutions of the KKT systems corresponding to (5.1) and (5.1$_\text{s}$).

(*ii*) Show that GCQ/ACQ/MFCQ/LICQ is satisfied at a feasible (x, s) for (5.1$_\text{s}$) if the respective condition is satisfied at x for (5.1).

For which CQs can you show equivalence?

Homework Problem 12.2 (Generalized derivatives) 5 Points (*i*) Compute the Bouligand- and Clarke generalized derivatives for $f: \mathbb{R} \to \mathbb{R}$, $f(x) = |x|$ at every $x \in \mathbb{R}$.

(*ii*) Show that if $f: \mathbb{R}^n \to \mathbb{R}^m$ is Lipschitz continuous on some neighborhood of $x \in \mathbb{R}^n$, then the Bouligand generalized derivative $\partial_B f(x)$ and the Clarke generalized derivative $\partial f(x)$ are nonempty and compact. In addition, $\partial f(x)$ is convex.

Homework Problem 12.3 (Semismooth NCP functions) 6 Points

Show that

$$\Phi_{\min}(a,b) := \min\{a,b\} \qquad \text{“min” function,} \tag{11.8a}$$

$$\Phi_{\mathrm{FB}}(a,b) := \sqrt{a^2+b^2} - a - b \qquad \text{Fischer-Burmeister function (Fischer, 1992)} \tag{11.8b}$$

as functions from $\mathbb{R}^2 \to \mathbb{R}$

(*i*) are NCP functions (Definition 11.4).

(*ii*) are semismooth everywhere (Definition 11.7).

Homework Problem 12.4 (Reduced reformulation of the semismooth Newton step) 3 Points

Show that the semismooth Newton step (in abbreviated notation), cf. Equation (11.15):

$$\begin{bmatrix} H & -\mathrm{Id} & B^\intercal \\ D_{\mathcal{A}} & D_{\mathcal{I}} & 0 \\ B & 0 & 0 \end{bmatrix} \begin{pmatrix} d \\ \mu \\ \lambda \end{pmatrix} = \begin{pmatrix} b \\ D_{\mathcal{A}}\ell \\ c \end{pmatrix}$$

can be transferred by using selection matrices

$$Z_{\mathcal{A}} := \text{rows of Id} \in \mathbb{R}^{n\times n} \text{ pertaining to active indices}$$
$$Z_{\mathcal{I}} := \text{rows of Id} \in \mathbb{R}^{n\times n} \text{ pertaining to inactive indices}$$

and subvectors $d_{\mathcal{A}} = Z_{\mathcal{A}} d$, $d_{\mathcal{I}} = Z_{\mathcal{I}} d$, $\mu_{\mathcal{A}} = Z_{\mathcal{A}}\mu$, $\mu_{\mathcal{I}} = Z_{\mathcal{I}}\mu$ into the equivalent reduced problem, cf. Equation (11.16):

$$\begin{bmatrix} Z_{\mathcal{I}} H Z_{\mathcal{I}}^\intercal & Z_{\mathcal{I}} B^\intercal \\ B Z_{\mathcal{I}}^\intercal & 0 \end{bmatrix} \begin{pmatrix} d_{\mathcal{I}} \\ \lambda \end{pmatrix} = \begin{pmatrix} Z_{\mathcal{I}}(b - H D_{\mathcal{A}}\ell) \\ c - B D_{\mathcal{A}}\ell \end{pmatrix}.$$

Please submit your solutions as a single pdf and an archive of programs via moodle.

References

Fischer, A. (1992). "A special Newton-type optimization method". *Optimization. A Journal of Mathematical Programming and Operations Research* 24.3-4, pp. 269–284. DOI: 10.1080/02331939208843795.