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Exercise 10

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Homework Problem 10.1 (Comparing the Strength of CQs) 6 Points

From the lecture notes, we know that

$$\boxed{\text{LICQ}} \xRightarrow{\text{Lemma 6.17}} \boxed{\text{MFCQ}} \xRightarrow{\text{Corollary 6.14}} \boxed{\text{ACQ}} \xRightarrow{\text{Definition 6.6}} \boxed{\text{GCQ}}. \tag{6.15}$$

Show that generally

$$\boxed{\text{LICQ}} \overset{(\text{P3})}{\not\Leftarrow} \boxed{\text{MFCQ}} \overset{(\text{P2})}{\not\Leftarrow} \boxed{\text{ACQ}} \overset{(\text{P1})}{\not\Leftarrow} \boxed{\text{GCQ}}$$

by investigating the following problems P1 to P3 at $x^* = (0, 0)^\intercal$:

$$\left.\begin{aligned} \text{Minimize} \quad & f(x) \qquad \text{where } x \in \mathbb{R}^2 \\ \text{subject to} \quad & x_1 \le 0 \\ & x_2 \le 0 \\ & x_1 x_2 = 0 \end{aligned}\right\} \tag{P1}$$

$$\left.\begin{aligned} \text{Minimize} \quad & f(x) \qquad \text{where } x \in \mathbb{R}^2 \\ \text{subject to} \quad & q(x_1) - x_2 \le 0 \\ & q(x_1) + x_2 \le 0 \end{aligned}\right\} \quad \text{for} \quad q(x_1) := \begin{cases} (x_1+1)^2, & x_1 < -1, \\ 0, & -1 \le x_1 \le 1, \\ (x_1-1)^2, & x_1 > 1, \end{cases} \tag{P2}$$

$$\left.\begin{aligned} \text{Minimize} \quad & f(x) \qquad \text{where } x \in \mathbb{R}^2 \\ \text{subject to} \quad & -x_1^3 - x_2 \le 0 \\ & -x_2 \le 0 \end{aligned}\right\} \tag{P3}$$

Homework Problem 10.2 (Finding Solutions using First and Second Order Information) 6 Points

Consider the problem

$$\left.\begin{aligned} \text{Maximize} \quad & -(x_1-2)^2 - 2(x_2-1)^2 \quad \text{where } x \in \mathbb{R}^2 \\ \text{subject to} \quad & x_1 + 4x_2 \le 3 \\ \text{and} \quad & x_1 \ge x_2 \end{aligned}\right\}$$

Determine, which admissible points satisfy a constraint qualification (ACQ/GCQ/MFCQ/LICQ) and use first and second order information to compute all stationary points and solve the problem, i. e., find all optima and explain why they are local and/or global solutions.

Homework Problem 10.3 (Solvability and global solutions of equality constrained QPs) 6 Points

Prove Lemma 9.2 of the lecture notes, i. e., the following statements for the quadratic problem

$$\begin{aligned} \text{Minimize} \quad & \mathcal{L}(\overline{x}, \overline{\lambda}) + \mathcal{L}_x(\overline{x}, \overline{\lambda})\, d + \frac{1}{2} d^\top \mathcal{L}_{xx}(\overline{x}, \overline{\lambda})\, d, \quad \text{where } d \in \mathbb{R}^n \\ \text{subject to} \quad & h(\overline{x}) + h'(\overline{x})\, d = 0. \end{aligned} \tag{9.1}$$

(i) Suppose that the linear system $h(\overline{x}) + h'(\overline{x})\, d = 0$ is solvable, and that d_{part} is some particular solution. Suppose, moreover, that the reduced Hessian $Z^\top \mathcal{L}_{xx}(\overline{x}, \overline{\lambda})\, Z$ is positive semidefinite. Then the objective in the reduced QP (9.3) is convex. In this case, the following are equivalent:

(a) The QP (9.1) possesses at least one (global) minimizer.

(b) The QP (9.1) is neither unbounded nor infeasible.

(c) The KKT conditions (9.2) are solvable.

(d) The reduced QP (9.3) possesses at least one (global) minimizer.

(e) The reduced QP (9.3) is not unbounded.

(f) The first-order optimality condition (9.4) is solvable.

The global minimizers of (9.1) are precisely the KKT points, i. e., the d-components of solutions (d, λ) to the KKT system (9.2).

(*ii*) Suppose that the linear system $h(\overline{x}) + h'(\overline{x})\, d = 0$ is solvable, and that d_{part} is some particular solution. Suppose now that the reduced Hessian $Z^\intercal \mathcal{L}_{xx}(\overline{x}, \overline{\lambda})\, Z$ is not positive semidefinite. Then the QP (9.1) and the reduced QP (9.3) are unbounded.

(*iii*) Suppose that the linear system $h(\overline{x}) + h'(\overline{x})\, d = 0$ is not solvable. Then the QP (9.1) is infeasible and the reduced QP cannot be formulated for lack of a particular solution d_{part}.

Homework Problem 10.4 (LICQ is equivalent to a unique Lagrange multiplier for certain QPs) 3 Points

Consider the (affine linearly) equality constrained quadratic optimization problem of the form

$$\begin{aligned} &\text{Minimize} \quad \frac{1}{2}x^\intercal A x + b^\intercal x + c, \quad \text{where } x \in \mathbb{R}^n \\ &\text{subject to} \quad Cx = d \end{aligned} \tag{0.1}$$

for symmetric $A \in \mathbb{R}^{n \times n}$, $b \in \mathbb{R}^n$, $c \in \mathbb{R}$ and $C \in \mathbb{R}^{n \times n_{\text{eq}}}$, $d \in \mathbb{R}^{n_{\text{eq}}}$ and let x^* be a KKT-point of (0.1).

Show that the set $\Lambda(x^*)$ of Lagrange multipliers corresponding to x^* is a singleton if and only if the LICQ is satisfied at x^*.

Note: This proves the second set of equivalences in Equation (9.5).

Please submit your solutions as a single pdf and an archive of programs via moodle.