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Exercise 10 - Solution

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Homework Problem 10.1 (Comparing the Strength of CQs) 6 Points

From the lecture notes, we know that

$$\boxed{\text{LICQ}} \xRightarrow{\text{Lemma 6.17}} \boxed{\text{MFCQ}} \xRightarrow{\text{Corollary 6.14}} \boxed{\text{ACQ}} \xRightarrow{\text{Definition 6.6}} \boxed{\text{GCQ}}. \tag{6.15}$$

Show that generally

$$\boxed{\text{LICQ}} \overset{(\text{P3})}{\not\Leftarrow} \boxed{\text{MFCQ}} \overset{(\text{P2})}{\not\Leftarrow} \boxed{\text{ACQ}} \overset{(\text{P1})}{\not\Leftarrow} \boxed{\text{GCQ}}$$

by investigating the following problems P1 to P3 at $x^* = (0,0)^\intercal$:

$$\left.\begin{aligned} \text{Minimize} \quad & f(x) \qquad \text{where } x \in \mathbb{R}^2 \\ \text{subject to} \quad & x_1 \le 0 \\ & x_2 \le 0 \\ & x_1 x_2 = 0 \end{aligned}\right\} \tag{P1}$$

$$\left.\begin{aligned} \text{Minimize} \quad & f(x) \qquad \text{where } x \in \mathbb{R}^2 \\ \text{subject to} \quad & q(x_1) - x_2 \le 0 \\ & q(x_1) + x_2 \le 0 \end{aligned}\right\} \quad \text{for} \quad q(x_1) := \begin{cases} (x_1+1)^2, & x_1 < -1, \\ 0, & -1 \le x_1 \le 1, \\ (x_1-1)^2, & x_1 > 1, \end{cases} \tag{P2}$$

$$\left.\begin{aligned} \text{Minimize} \quad & f(x) \qquad \text{where } x \in \mathbb{R}^2 \\ \text{subject to} \quad & -x_1^3 - x_2 \le 0 \\ & -x_2 \le 0 \end{aligned}\right\} \tag{P3}$$

Solution.

(i) We start out showing that LICQ is not satisfied at the origin in problem P3, but MFCQ is. **Note:** LICQ requires that the active constraints do not carry the same tangent information while MFCQ only requires that there is a direction pointing into the interior of all active inequality constraints' feasibility regions. The example was constructed using this information.

At the point in question, both inequality constraints of problem P3 are active and we have

$$\begin{aligned} g_1'(x) &= \left(-3x_1^2, -1\right) &&\implies g_1'(x^*) = (0, -1) \\ g_2'(x) &= (0, -1) &&\implies g_2'(x^*) = (0, -1) \end{aligned}$$

so LICQ in fact does not hold. However, as there are no equality constraints, the direction $d = (0, 1)^\mathsf{T}$ shows that MFCQ in fact holds.

(ii) Next up, we show that MFCQ is violated at the origin for problem P2, but ACQ holds. **Note:** MFCQ requires that there is a direction pointing into the interior of all active inequality constraints' feasibility regions, so this prehibits that two active, countering inequality constraints collaps to yield an equality constraint. ACQ only requires that the tangent information and the linearized information match up. The example was constructed using this information. (2 Points)

At the point in question, both inequality constraints of problem P2 are active. Note that q is continuously differentiable with

$$q'(x_1) = \begin{cases} 2(x_1 + 1), & x_1 < -1, \\ 0, & -1 \le x_1 \le 1, \\ 2(x_1 - 1), & x_1 > 1, \end{cases} \tag{0.1}$$

and we have

$$\begin{aligned} g_1'(x) &= (q'(x_1), -1) &&\implies g_1'(x^*) = (0, -1) \\ g_2'(x) &= (q'(x_1), 1) &&\implies g_2'(x^*) = (0, 1) \end{aligned}$$

so $g_1'(x^*)d = d_2 = -g_2'(x^*)d$ for any $d \in \mathbb{R}^2$ cannot be simultaneously greater and less than 0, so MFCQ is violated. However, we have that

$$\mathcal{T}_F(x^*) = \mathbb{R} \times \{0\} = \mathcal{T}_F^{\text{lin}}(x^*),$$

so ACQ is satisfied. (2 Points)

(iii) Finally, we show that ACQ is violated in problem P1, but GCQ is satisfied. **Note:** ACQ needs the tangent and linearizing information/sets to match up while GCQ only requires that their polar cones coincide, i. e., that they look "the same on the outside" to linear functionals applied to them. This is a property of complementarity constrained problems such as the one used in this example.

For the feasible set $F = \{x \in \mathbb{R}^2 \mid x_1, x_2 \geq 0,\ x_1 x_2 = 0\}$, with $x^* = (0, 0) \in F$, we have that

$$\begin{aligned} g_1'(x) &= (1, 0) && \Longrightarrow\ g_1'(x^*) = (1, 0) \\ g_2'(x) &= (0, 1) && \Longrightarrow\ g_2'(x^*) = (0, 1) \\ h'(x) &= (x_2, x_1) && \Longrightarrow\ h'(x^*) = (0, 0) \end{aligned}$$

we easily obtain the cones

$$\begin{aligned} \mathcal{T}_F(x^*) &= F \\ \mathcal{T}_F^{\text{lin}}(x^*) &= \mathbb{R}^2_{\leq} = \operatorname{conv}(F). \end{aligned}$$

Accordingly, the cones don't match because the linearizing cone fills in the entire lower left quadrant. The set's boundaries still coincide though, and we obtain

$$\mathcal{T}_F(x^*)^\circ = \mathbb{R}^2_{\geq} = \mathcal{T}_F^{\text{lin}}(x^*)^\circ,$$

so GCQ is satisfied. (2 Points)

Homework Problem 10.2 (Finding Solutions using First and Second Order Information) 6 Points

Consider the problem

$$\left.\begin{aligned} \text{Maximize} \quad & -(x_1 - 2)^2 - 2(x_2 - 1)^2 \quad \text{where } x \in \mathbb{R}^2 \\ \text{subject to} \quad & x_1 + 4x_2 \leq 3 \\ \text{and} \quad & x_1 \geq x_2 \end{aligned}\right\}$$

Determine, which admissible points satisfy a constraint qualification (ACQ/GCQ/MFCQ/LICQ) and use first and second order information to compute all stationary points and solve the problem, i. e., find all optima and explain why they are local and/or global solutions.

Solution.

We rewrite the problem as a minimization problem by considering the negative of the cost function and consequently obtain

$$f(x) = (x_1 - 2)^2 + 2(x_2 - 1)^2 \qquad g_1(x) = x_1 + 4x_2 - 3 \qquad g_2(x) = x_2 - x_1,$$
$$f'(x) = (2x_1 - 4, 4x_2 - 4) \qquad g_1'(x) = (1, 4) \qquad g_2'(x) = (-1, 1).$$

Since $g_i'(x) \neq 0$ for all x and $i = 1, 2$ and both are obviously always linear independent, the LICQ is satisfied at all feasible x. (1 Point)

The corresponding KKT system for a multiplier $\mu \geq 0 \in \mathbb{R}^2$ and feasible $x = (x_1, x_2)^\top$ is

$$\begin{aligned} 2x_1 - 4 + \mu_1 - \mu_2 &= 0 \\ 4x_2 - 4 + 4\mu_1 + \mu_2 &= 0 \\ \mu_1(x_1 + 4x_2 - 3) &= 0 \\ \mu_2(x_2 - x_1) &= 0. \end{aligned}$$

Due to the complementarity conditions, there are four cases:

(i) Both constraints are inactive ($\mu_1 = 0,\ \mu_2 = 0$): The KKT system will then reduce to

$$\begin{aligned} 2x_1 - 4 &= 0 \\ 4x_2 - 4 &= 0 \end{aligned}$$

with the solution $x^* = (x_1^*, x_2^*) = (2, 1)$, which violates the first constraint so there is no stationary point, and hence no solution, for this case. (1 Point)

(ii) The first constraint is inactive and the second is active ($\mu_1 = 0$ and $x_1 = x_2$): Here the KKT system is

$$\begin{aligned} 2x_1 - 4 - \mu_2 &= 0, \\ 4x_2 - 4 + \mu_2 &= 0, \\ x_1 &= x_2. \end{aligned}$$

Adding the first two equations and using the active second constraint, we obtain the solution $\left(\frac{4}{3}, \frac{4}{3}\right)$ with $\mu_2 = -\frac{4}{3}$, which violates the nonnegativity condition so there is no stationary point, and hence no solution, for this case. (1 Point)

(iii) The second constraint is inactive and the first constraint is active ($\mu_2 = 0$ and $x_1 + 4x_2 = 3$): Here, the KKT system is

$$\begin{aligned} 2x_1 - 4 + \mu_1 &= 0 \\ 4x_2 - 4 + 4\mu_1 &= 0 \\ x_1 + 4x_2 &= 3. \end{aligned}$$

Adding the second equation to -4 times the first equation, we get

$$\begin{aligned} -8x_1 + 4x_2 &= -12 \\ x_1 + 4x_2 &= 3 \end{aligned}$$

with the unique solution $\left(\frac{5}{3}, \frac{1}{3}\right)$ and $\mu_1 = \frac{2}{3}$. This is a stationary point as x is feasible an μ is nonnegative. (1 Point)

(iv) Both constraints are active ($x_1 + 4x_2 = 3$ and $x_1 = x_2$): Here the KKT system is

$$\begin{aligned} 2x_1 - 4 + \mu_1 - \mu_2 &= 0 \\ 4x_2 - 4 + 4\mu_1 + \mu_2 &= 0 \\ x_1 + 4x_2 - 3 &= 0 \\ x_2 - x_1 &= 0 \end{aligned}$$

with the solution $\left(\frac{3}{5}, \frac{3}{5}\right)$ and $\mu_1 = \frac{22}{25}$, $\mu_2 = -\frac{48}{25}$, which violates the nonnegativity constaint for μ, so there is no stationary point, and hence no solution, for this case. (1 Point)

The feasible set is not compact, so we need second order information to check the optimality of the stationary point. We compute

$$\mathcal{L}_{xx}(x, \mu) = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}$$

which is positive definite. Hence, the solution to the problem is a local maximizer of the original (maximization) problem at $\left(\frac{5}{3}, \frac{1}{3}\right)$, with functional value $-J\left(\frac{5}{3}, \frac{1}{3}\right) = 1$. Due to convexity of the cost functional and the feasible set, the local maximizer is the only maximizer and is a global one. (1 Point)

Homework Problem 10.3 (Solvability and global solutions of equality constrained QPs) 6 Points

Prove Lemma 9.2 of the lecture notes, i. e., the following statements for the quadratic problem

$$\begin{aligned} &\text{Minimize} \quad \mathcal{L}(\overline{x}, \overline{\lambda}) + \mathcal{L}_x(\overline{x}, \overline{\lambda})\, d + \frac{1}{2}\, d^\intercal \mathcal{L}_{xx}(\overline{x}, \overline{\lambda})\, d, \quad \text{where } d \in \mathbb{R}^n \\ &\text{subject to} \quad h(\overline{x}) + h'(\overline{x})\, d = 0. \end{aligned} \tag{9.1}$$

(i) Suppose that the linear system $h(\overline{x}) + h'(\overline{x})\, d = 0$ is solvable, and that d_{part} is some particular solution. Suppose, moreover, that the reduced Hessian $Z^\intercal \mathcal{L}_{xx}(\overline{x}, \overline{\lambda})\, Z$ is positive semidefinite. Then the objective in the reduced QP (9.3) is convex. In this case, the following are equivalent:

(a) The QP (9.1) possesses at least one (global) minimizer.

(b) The QP (9.1) is neither unbounded nor infeasible.

(c) The KKT conditions (9.2) are solvable.

(d) The reduced QP (9.3) possesses at least one (global) minimizer.

(e) The reduced QP (9.3) is not unbounded.

(f) The first-order optimality condition (9.4) is solvable.

The global minimizers of (9.1) are precisely the KKT points, i. e., the d-components of solutions (d, λ) to the KKT system (9.2).

(ii) Suppose that the linear system $h(\overline{x}) + h'(\overline{x})\, d = 0$ is solvable, and that d_{part} is some particular solution. Suppose now that the reduced Hessian $Z^\intercal \mathcal{L}_{xx}(\overline{x}, \overline{\lambda})\, Z$ is not positive semidefinite. Then the QP (9.1) and the reduced QP (9.3) are unbounded.

(iii) Suppose that the linear system $h(\overline{x}) + h'(\overline{x})\, d = 0$ is not solvable. Then the QP (9.1) is infeasible and the reduced QP cannot be formulated for lack of a particular solution d_{part}.

Solution.

(i) Clearly, the reformulation in the lecture notes that used the representation of feasible d as

$$d = d_{\text{part}} + Z\, y$$

shows that the original constrained and the reduced problems are equivalent.

Accordingly, we immediately see that Statement (a) $\Leftrightarrow$ Statement (d) and Statement (b) $\Leftrightarrow$ Statement (e).

Additionally, Lemma 3.1 immediately applies to the reduced problem

$$\text{Minimize} \quad \left[\mathcal{L}_x(\overline{x}, \overline{\lambda}) + d_{\text{part}}^\intercal \mathcal{L}_{xx}(\overline{x}, \overline{\lambda})\right] Z\, y + \frac{1}{2}\, y^\intercal Z^\intercal \mathcal{L}_{xx}(\overline{x}, \overline{\lambda})\, Z\, y, \quad \text{where } y \in \mathbb{R}^{n-r}. \tag{9.3}$$

so we obtain that Statement (d) $\Leftrightarrow$ Statement (e) $\Leftrightarrow$ Statement (f) and that the global minimizers are precisely the solutions to the reduced first order optimality conditions (9.2).

Accordingly, we only need to tie in Statement (c). Since the constraints are affine, we know that ACQ holds at every feasible point (see homework problem 9.2), hence every (local) minimizer is a KKT point making up the d components of the KKT conditions. We would like to show the reverse statement using Theorem 6.19, but that is formulated using the convexity of the cost functional everywhere (which we only have on the feasible set). One could adjust the proof to deal with the weaker assumption but would have to reformulate the lower linear approximation result for convex functions as well, so instead, we proceed to notice that if (d, λ) is a KKT point, then d is feasible and accordingly

$$d = d_{\text{part}} + Zy$$

Plugging this into the first line of the block system yields that

$$\mathcal{L}_{xx}(\overline{x}, \overline{\lambda})(d_{\text{part}} + Zy) + h'(\overline{x})^\intercal \lambda = -\nabla_x \mathcal{L}(\overline{x}, \overline{\lambda})$$

and, since Z spans the kernel of $h'(\overline{x})$, multiplying this line by $Z^\intercal$ yields the first order system (9.4), meaning that y is a global minimizer to the reduced QP and therefore d is a global minimizer for the original QP.

(ii) Both the QP and the reduced QP are equivalent. The unboundedness is easiest to see for the reduced problem. Since $Z^\intercal \mathcal{L}_{xx} Z$ is indefinite, it has at least one negative eigenvalue $\nu < 0$ with corresponding eigenvector $\tilde{y} \neq 0$, i. e., a direction of negative curvature in the feasible set. Accordingly:

$$\begin{aligned}\left[\mathcal{L}_x(\overline{x}, \overline{\lambda}) + d_{\text{part}}^\intercal \mathcal{L}_{xx}(\overline{x}, \overline{\lambda})\right] Z t\tilde{y} + \frac{1}{2} t\tilde{y}^\intercal Z^\intercal \mathcal{L}_{xx}(\overline{x}, \overline{\lambda})\, Z t\tilde{y} &= \\ \frac{1}{2} t\nu \|\tilde{y}\|_2^2 + \left[\mathcal{L}_x(\overline{x}, \overline{\lambda}) + d_{\text{part}}^\intercal \mathcal{L}_{xx}(\overline{x}, \overline{\lambda})\right] Z t\tilde{y}& \\ \xrightarrow{t\to\infty} -\infty&\end{aligned}$$

meaning that we can reach arbitrarily small function values by feasible points.

(iii) This is obvious.

(6 Points)

Homework Problem 10.4 (LICQ is equivalent to a unique Lagrange multiplier for certain QPs) 3 Points

Consider the (affine linearly) equality constrained quadratic optimization problem of the form

$$\begin{aligned} \text{Minimize} \quad & \frac{1}{2} x^\top A x + b^\top x + c, \quad \text{where } x \in \mathbb{R}^n \\ \text{subject to} \quad & Cx = d \end{aligned} \tag{0.2}$$

for symmetric $A \in \mathbb{R}^{n \times n}$, $b \in \mathbb{R}^n$, $c \in \mathbb{R}$ and $C \in \mathbb{R}^{n \times n_{\text{eq}}}$, $d \in \mathbb{R}^{n_{\text{eq}}}$ and let x^* be a KKT-point of (0.2).

Show that the set $\Lambda(x^*)$ of Lagrange multipliers corresponding to x^* is a singleton if and only if the LICQ is satisfied at x^*.

Note: This proves the second set of equivalences in Equation (9.5).

Solution.

The KKT system of (0.2) for $x \in \mathbb{R}^n$, $\lambda \in \mathbb{R}^{n_{\text{eq}}}$ is

$$\begin{aligned} Ax + b + C^\top \lambda &= 0 \\ Cx &= d \end{aligned} \tag{0.3}$$

and by assumption, there exists a $\lambda^* \in \mathbb{R}^{n_{\text{eq}}}$ corresponding to x^* such that (x^*, λ^*) solves the system (0.3). Due to the first line of the same system, we know that

$$\Lambda(x^*) = \lambda^* + \ker C^\top.$$

Additionally, $\ker C^\top = \{0\}$ if and only if $C^\top$ has full column rank, which is exactly the LICQ at x^*. (However, note that for this affine constraint, LICQ is satisfied at all feasible points or at none of them). Otherwise the kernel is a nontrivial subspace of positive dimension, meaning $\Lambda(x^*)$ is a nontrivial affine subspace (and therefore not compact, which we were to expect because LICQ is equivalent to MFCQ for equality constrained problems and MFCQ is equivalent to compact multiplier sets).

(3 Points)

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