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Exercise 9

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Homework Problem 9.1 (Lin. Cone and CQs Depend on Description of Feasible Set) 3 Points

The optimization problems

$$\begin{aligned} \text{Minimiere} \quad & f(x) \quad \text{über } x \in \mathbb{R} \\ \text{unter} \quad & x = 0 \end{aligned} \tag{P_1}$$

and

$$\begin{aligned} \text{Minimiere} \quad & f(x) \quad \text{über } x \in \mathbb{R} \\ \text{unter} \quad & x^2 = 0 \end{aligned} \tag{P_2}$$

for any $f \in C^1(\mathbb{R})$ have their obvious solution (because sole feasible point) at $x^* = 0$.

Show that the Abadie and Guignard constraint qualifications are satisfied at $x^* = 0$ for (P_1) but not (P_2).

Homework Problem 9.2 (ACQ for Problems with Affine Constraints) 6 Points

Consider

$$F := \left\{ x \in \mathbb{R}^n \,\middle|\, g_i(x) \le 0 \text{ for all } i = 1, \ldots, n_{\text{ineq}},\; h_j(x) = 0 \text{ for all } j = 1, \ldots, n_{\text{eq}} \right\} \tag{5.2}$$

and

$$F^{\text{lin}}(x) = \left\{ y \in \mathbb{R}^n \,\middle|\, \begin{array}{ll} g_i(x) + g_i'(x)\,(y - x) \le 0 & \text{for all } i = 1, \ldots, n_{\text{ineq}} \\ h_j(x) + h_j'(x)\,(y - x) = 0 & \text{for all } j = 1, \ldots, n_{\text{eq}} \end{array} \right\}$$

for $x \in F$.

(i) Show that $\mathcal{T}_F^{\text{lin}}(x) = \mathcal{T}_{F^{\text{lin}}(x)}(x)$ for $x \in F$. (Remark 5.6 Statement (i))

(*ii*) Show that $\mathcal{T}_F^{\text{lin}}(x)$ is a closed convex cone. (Remark 5.6 Statement (*ii*))

(*iii*) Prove Theorem 6.9 by showing that the Abadie CQ holds at any feasible point of problems of the form

$$\left.\begin{aligned} \text{Minimize} \quad & f(x) \qquad \text{where } x \in \mathbb{R}^n \\ \text{subject to} \quad & A_{\text{ineq}}\, x \le b_{\text{ineq}} \\ \text{and} \quad & A_{\text{eq}}\, x = b_{\text{eq}} \end{aligned}\right\} \tag{6.10}$$

Homework Problem 9.3 (Alternative formulation of the second MFCQ condition) 6 Points

Prove Lemma 6.11, i.e. show that the following three statements are equivalent for $x \in F$.

(*ii*) There exists a vector $d \in \mathbb{R}^n$ such that

$$\begin{aligned} g_i'(x)\, d &< 0 \quad \text{for all } i \in \mathcal{A}(x), \\ h_j'(x)\, d &= 0 \quad \text{for all } j = 1, \ldots, n_{\text{eq}}. \end{aligned} \tag{0.1}$$

(*iii*) There exists a vector $d \in \mathbb{R}^n$ such that

$$\begin{aligned} g(x) + g'(x)\, d &< 0, \\ h(x) + h'(x)\, d &= 0. \end{aligned} \tag{0.2}$$

(*iv*) There exists a vector $d \in \mathbb{R}^n$ such that

$$\begin{aligned} g_i'(x)\, d &\le -1 \qquad \text{for all } i \in \mathcal{A}(x), \\ h_j'(x)\, d &= 0 \qquad \text{for all } j = 1, \ldots, n_{\text{eq}}. \end{aligned} \tag{0.3}$$

Homework Problem 9.4 (Multiplier Compactness is Equivalent to MFCQ) 6 Points

(*i*) Use Farkas' Lemma (Lemma 5.11 in the lecture notes) to show that for $A \in \mathbb{R}^{p \times n}$ and $B \in \mathbb{R}^{n_{\text{eq}} \times n}$ with $\operatorname{rank}(B) = n_{\text{eq}}$ and $p \le n_{\text{ineq}}$ either the system

$$Ad < 0, \quad Bd = 0 \tag{0.4}$$

has a solution $d \in \mathbb{R}^n$ or

$$A^\mathsf{T} \mu + B^\mathsf{T} \lambda = 0 \tag{0.5}$$

has a solution $(\mu, \lambda) \neq 0$ with $\mu \geq 0$.

Hint: Start with the existence of a nontrivial solution to (0.5). Focus the nontriviality on μ. Transform the conditions $\mu \neq 0, \mu \geq 0$ into a linear condition with a sign condition using a normalization step with respect to $\|\cdot\|_1$. Split λ into its positive and negative part. Apply Farkas' Lemma. Success.

(ii) Let (x^*, λ^*, μ^*) be a KKT-point of (5.1). Show that MFCQ is satisfied at x^* if and only if the set of Lagrange multipliers that solve the KKT system for x^* is compact.

Please submit your solutions as a single pdf and an archive of programs via moodle.