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Exercise 8

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Homework Problem 8.1 (Two-Loop Recursion for Inverse BFGS Update) 6 Points

Show that Algorithm 4.53 in fact computes the action of the inverse BFGS updated matrix $B^{(k)}_{\text{BFGS}}$.

Homework Problem 8.2 (Examples for Tangent-, Linearizing and Normal Cones) 5 Points

Consider the feasible set

$$F := \left\{ x \in \mathbb{R}^n \,\middle|\, g_i(x) \le 0 \text{ for all } i = 1, \dots, n_{\text{ineq}}, \; h_j(x) = 0 \text{ for all } j = 1, \dots, n_{\text{eq}} \right\} \tag{5.2}$$

without any equality restrictions h and with the inequality constraints $g\colon \mathbb{R}^3 \to \mathbb{R}^4$ defined by

$$g(x) = \begin{pmatrix} (x_1 - 1)^2 + x_2^2 - 1 \\ (x_1 - 3)^2 + x_2^2 - 1 \\ x_3 + 1 \\ -x_3 - 2 \end{pmatrix} \qquad \text{at} \qquad x^* = (2,\ 0,\ -1)^\intercal \in F.$$

Find the set of active indices $\mathcal{A}(x^*)$, an explicit representation of F, the tangent cone $\mathcal{T}_F(x^*)$, the **normal cone** $\mathcal{T}_F(x^*)^\circ$ and the linearizing cone $\mathcal{T}_F^{\text{lin}}(x^*)$ and sketch F and the cones.

Homework Problem 8.3 (Linearizing Cone Depends on Description of Feasible Set) 2 Points

Consider the sets

$$F^{(1)} := \left\{ x \in \mathbb{R}^2 \,\middle|\, \begin{pmatrix} -x_1 - 1 \\ x_1 - 1 \end{pmatrix} \le 0,\ x_2 = 0 \right\}, \quad \text{and} \quad F^{(2)} := \left\{ x \in \mathbb{R}^2 \,\middle|\, \begin{pmatrix} x_2 - (x_1 + 1)^3 \\ x_1 - 1 \end{pmatrix} \le 0,\ x_2 = 0 \right\}.$$

Find an explicit description of the sets $F^{(1/2)}$ and compare the linearizing cones $\mathcal{T}^{\text{lin}}_{F^{(1/2)}}(x)$ at $x^* = (-1, 0)$.

Homework Problem 8.4 (Examples and Properties of Polar Cones) 4 Points

(i) Prove Lemma 5.9 of the lecture notes, i. e., for arbitrary sets $M, M_1, M_2 \subseteq \mathbb{R}^n$ the statements

(a) M° is a closed convex cone.

(b) $M_1 \subseteq M_2$ implies $M_2^\circ \subseteq M_1^\circ$.

(ii) Verify the claimed forms of the polar cones in Example 5.10, i. e., the following:

(a) Suppose that A is an affine subspace of $\mathbb{R}^n$ of the form $A = U + \{\bar{x}\}$. Then $A^\circ = \{\bar{x}\}^\circ \cap U^\perp$.

(b) In the absence of inequality constraints, the polar of the linearizing cone $\mathcal{T}_F^{\text{lin}}(x)$ for $x \in F$ has the representation

$$\begin{aligned} \mathcal{T}_F^{\text{lin}}(x)^\circ &= \{s \in \mathbb{R}^n \mid s \text{ is some linear combination of } h_j'(x)^\intercal,\ j = 1, \ldots, n_{\text{eq}}\} \\ &= \operatorname{range} h'(x)^\intercal. \end{aligned}$$

(c) Let $N := (\mathbb{R}_{\geq 0})^n$ denote the non-negative orthant in $\mathbb{R}^n$. Then $N^\circ = (\mathbb{R}_{\leq 0})^n$ is the non-positive orthant.

Please submit your solutions as a single pdf and an archive of programs via moodle.