Exercise 7

Date issued: 27th May 2024
Date due: 4th June 2024

Homework Problem 7.1 (The truncated CG method generates descent directions) 7 Points

Prove the statements from Lemma 4.42.

How can property Statement *(iii)* be interpreted, when the truncated CG method is applied to the current Newton system $f''(x^{(k)})\, d^{(k)} = -\nabla f(x^{(k)})$?

Homework Problem 7.2 (Truncated Newton CG) 6 Points

Implement the truncated Newton-CG method (Algorithm 4.44 with Algorithm 4.41), apply it for Rosenbrock's and/or Himmelblau's functions and compare its performance with the exact globalized Newton method.

Homework Problem 7.3 (Inverse BFGS and DFP Updates) 6 Points

Derive the inverse BFGS and DFP update formulas

$$\Psi_{\text{BFGS}}(B, s, y) = (\text{Id} - \rho\, s\, y^\intercal)\, B\, (\text{Id} - \rho\, y\, s^\intercal) + \rho\, s\, s^\intercal, \tag{4.60}$$

$$\Psi_{\text{DFP}}(B, s, y) = B - \frac{B\, y\, y^\intercal B}{y^\intercal B\, y} + \rho\, s\, s^\intercal \tag{4.59}$$

using the Sherman-Morrison-Woodbury formula from Lemma 4.50.

Homework Problem 7.4 (Affine-Invariance of BFGS/DFP-updated quasi Newton) 6 Points

Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be differentiable, $A \in \mathbb{R}^{n\times n}$ be invertible, $b \in \mathbb{R}^n$ and $g(y) \coloneqq f(Ay + b)$.

(i) Let the sequences $\left(x^{(k)}\right)$ and $\left(y^{(k)}\right)$ be generated by applying full quasi Newton steps as in

$$
\begin{aligned}
x^{(k+1)} &= x^{(k)} - H_f^{(k)^{-1}} f'(x^{(k)})^\intercal && \text{from } x^{(0)} = Ay^{(0)} + b && \text{with } H_f^{(0)} \text{ s. p. d.} \\
y^{(k+1)} &= y^{(k)} - H_g^{(k)^{-1}} g'(y^{(k)})^\intercal && \text{from } y^{(0)} && \text{with } H_g^{(0)} = A^\intercal H_f^{(0)} A.
\end{aligned}
$$

Show that $x^{(k)} = Ay^{(k)} + b$ for all $k \in \mathbb{N}$, when the BFGS or the DFP update are applied to update the model Hessians.

(ii) Let the sequences $\left(x^{(k)}\right)$ and $\left(y^{(k)}\right)$ be generated by applying full inverse quasi Newton steps as in

$$
\begin{aligned}
x^{(k+1)} &= x^{(k)} - B_f^{(k)} f'(x^{(k)})^\intercal && \text{from } x^{(0)} = Ay^{(0)} + b && \text{with } B_f^{(0)} \text{ s. p. d.} \\
y^{(k+1)} &= y^{(k)} - B_g^{(k)} g'(y^{(k)})^\intercal && \text{from } y^{(0)} && \text{with } B_g^{(0)} = A^{-1} B_f^{(0)} A^{-\intercal}.
\end{aligned}
$$

Show that $x^{(k)} = Ay^{(k)} + b$ for all $k \in \mathbb{N}$, when the inverse BFGS or the inverse DFP update are applied to update the inverse of the model Hessians.

Hint: You can save yourselves some work using the connection of the updates of the Hessians and their inverses.

Note: The restriction to unit step length scalings in this exercise is to keep the required notation slim(er). Since we know that the Armijo and the curvature condition are affine invariant as well, we don't loose invariance when applying step lengths satisfying these conditions.

Please submit your solutions as a single pdf and an archive of programs via moodle.