Exercise 3

Date issued: 29th April 2024
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Homework Problem 3.1 (Comparison of SD and CG Convergence) 5 Points

(i) Explain the differences and (dis-)advantages of the worst-case convergence results of the steepest descent and CG method for solving s. p. d. linear systems.

(ii) Plot the maximum number of iterations needed by the steepest descent and the CG method, respectively, (as guaranteed by their worst case convergence estimates), to produce a reduction of 10^{-6} in the error (relative to the initial error).

Homework Problem 3.2 (Any CG-history is better than no history.) 5 Points

Let $A \in \mathbb{R}^{n\times n}$ be s. p. d., $b \in \mathbb{R}^n$ and $c \in \mathbb{R}$. Further, let $x^{(k)}$ be an iterate of the CG method for minimizing $\phi(x) = \frac{1}{2}x^\intercal Ax - b^\intercal x + c$ and let $x^{(k+1)}$ be computed by an additional step of the CG method and $\widetilde{x}^{(k+1)}$ be computed by a steepest descent step with Cauchy stepsize starting from $x^{(k)}$.

Show that $\|x^{(k+1)} - x^*\|_A \le \|\widetilde{x}^{(k+1)} - x^*\|_A$.

Homework Problem 3.3 (Clustering of Eigenvalues) 6 Points

Let $A, M \in \mathbb{R}^{n\times n}$ be s. p. d., $b \in \mathbb{R}^n$ and $c \in \mathbb{R}$. Show that the M-preconditioned CG method will find the unique minimizer of $\phi(x) = \frac{1}{2}x^\intercal Ax - b^\intercal x + c$ in $k \le n$ iterations or less if any of the following conditions is satisfied.

(i) A has only k distinct generalized eigenvalues w.r.t. M;

(ii) The initial error $x^{(0)} - x^*$ is in the span of k distinct generalized eigenvectors of A w.r.t. M;

(*iii*) The initial M-gradient $M^{-1}r^{(0)}$ is in the span of k distinct generalized eigenvectors of A w.r.t. M.

Homework Problem 3.4 (M-Preconditioned CG for Solving s. p. d. Linear Systems) 6 Points

Implement the M-Preconditioned CG method for solving s. p. d. linear systems as outlined in Algorithm 3.17 of the lecture notes. Additionally include the option of turning conjugacy off (to return to the steepest descent method). Visualize the convergence speed of the method and compare the results to the steepest descent method for pseudo-randomized problems.

Please submit your solutions as a single pdf and an archive of programs via moodle.