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Exercise 1

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Homework Problem 1.1 (Optimality Condition Gap) 9 Points

Consider the optimization problem

$$\text{Minimize} \quad f(x) = \left(x_1 - x_2^2\right)\left(2x_1 - x_2^2\right) = 2x_1^2 - 3x_1x_2^2 + x_2^4 \quad \text{where } x \in \mathbb{R}^2.$$

(i) Show that the necessary optimality conditions of first and second order are satisfied at $(0,0)^\intercal$.

(ii) Show that $(0,0)^\intercal$ is a local minimizer for f along every straight line passing through $(0,0)$.

(iii) Show that $(0,0)^\intercal$ is not a local Minimizer of f on $\mathbb{R}^2$.

Homework Problem 1.2 (First Order Conditions are Sufficient for Convex Functions) 2 Points

Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a convex function that is differentiable at $x \in \mathbb{R}^n$ with $f'(x) = 0$. Show that x is a global minimizer of f.

Homework Problem 1.3 (Miscellaneous on Convergence Rates) 11 Points

(i) Explain why the definition of Q-quadratic convergence of a sequence requires the initial assumption that the sequence converges at all.

(ii) Show that Q-quadratic convergence implies Q-superlinear convergence which implies Q-linear convergence which implies convergence.

(iii) (a) Show that the notions of Q-linear, Q-superlinear and Q-quadratic convergence of a sequence imply their respective R-convergence counterparts.

(b) Give an example that shows that R-convergence of any kind of a sequence generally does not imply the corresponding Q-convergence.

(iv) (a) Let $\|\cdot\|_a, \|\cdot\|_b \colon \mathbb{R}^n \to \mathbb{R}$ be equivalent norms. Show that Q-superlinear resp. Q-quadratic convergence of a sequence w.r.t. $\|\cdot\|_a$ implies Q-superlinear resp. Q-quadratic convergence w.r.t. $\|\cdot\|_b$.

(b) Give an example that shows that the corresponding statement can not hold for Q-linear convergence. Does it hold for R-linear convergence?

Homework Problem 1.4

(Visualizing and Interpreting Convergence Rates) 7 Points

(i) For each of the following cases, give an example of a null sequence $\left(x^{(k)}\right)$ in $(\mathbb{R}, |\cdot|)$ that

(a) converges, but does not converge Q-linearly,
(b) converges Q-linearly, but does not converge Q-superlinear,
(c) converges Q-superlinearly, but does not converge Q-quadratically,
(d) converges Q-quadratically, but does not converge with higher order.

(ii) Explain what the Q-convergence rates of a sequence $x^k \to x^*$ will look like in a y-semi-logarithmic plot, i. e., when plotting the map $k \mapsto \ln |x^{(k)} - x^*|$.

(iii) Plot the distance to the limit for the sequences from task (i) over the iterations in a standard and a y-semi-logarithmic plot. What do you observe?

Please submit your solutions as a single pdf and an archive of programs via moodle.