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Exercise 13

Date issued: 10th July 2023
Date due: 18th July 2023

Homework Problem 13.1 (Differentiability of the ℓ_1-merit function) 2 Points

Verify that the directional derivative

$$\pi_1'(x;d) := \lim_{t \searrow 0} \frac{\pi_1(x + t\,d) - \pi_1(x)}{t}$$

of the ℓ_1-penalty part

$$\pi_1(x)\colon \mathbb{R}^n \to \mathbb{R}, \quad \pi_1(x) := \sum_{i=1}^{n_{\text{ineq}}} \max\{0, g_i(x)\} + \sum_{j=1}^{n_{\text{eq}}} |h_i(x)|$$

of the ℓ_1-merit function exists everywhere and is given by

$$\begin{aligned} \pi_1'(x;d) = \sum_{\substack{i=1\\ g_i(x)<0}}^{n_{\text{ineq}}} 0 \quad &+ \sum_{\substack{i=1\\ g_i(x)=0}}^{n_{\text{ineq}}} \max\{0, g_i'(x)\,d\} \quad + \sum_{\substack{i=1\\ g_i(x)>0}}^{n_{\text{ineq}}} g_i'(x)\,d \\ &+ \sum_{\substack{j=1\\ h_j(x)<0}}^{n_{\text{eq}}} -h_j'(x)\,d \quad + \sum_{\substack{j=1\\ h_j(x)=0}}^{n_{\text{eq}}} |h_j'(x)\,d| \quad + \sum_{\substack{j=1\\ h_j(x)>0}}^{n_{\text{eq}}} h_j'(x)\,d \end{aligned} \tag{14.2}$$

for $d \in \mathbb{R}^n$.

Homework Problem 13.2 (Penalty reformulation of infeasible SQP-subproblems) 1 Points

Show that the penalty reformulation

$$\begin{aligned}
\text{Minimize} \quad & \frac{1}{2} d^\top A\, d - b^\top d + \gamma \left[\mathbf{1}^\top v + \mathbf{1}^\top w + \mathbf{1}^\top t \right], \quad \text{where } (d, v, w, t) \in \mathbb{R}^n \times \mathbb{R}^{n_{\text{eq}}} \times \mathbb{R}^{n_{\text{eq}}} \times \mathbb{R}^{n_{\text{ineq}}} \\
\text{subject to} \quad & B_{\text{eq}}\, d - c_{\text{eq}} \quad = v - w \\
\text{and} \quad & B_{\text{ineq}}\, d - c_{\text{ineq}} \le t \\
\text{as well as} \quad & v \ge 0, \; w \ge 0, \; t \ge 0
\end{aligned} \tag{14.10}$$

of the SQP-subproblem-type problem

$$\begin{aligned}
\text{Minimize} \quad & \frac{1}{2} d^\top A\, d - b^\top d, \quad \text{where } d \in \mathbb{R}^n \\
\text{subject to} \quad & B_{\text{eq}}\, d - c_{\text{eq}} \quad = 0 \\
\text{and} \quad & B_{\text{ineq}}\, d - c_{\text{ineq}} \le 0
\end{aligned} \tag{14.9}$$

is always feasible.

Homework Problem 13.3 (Smoothness properties of exact penalty functions) 6 Points

Consider the constrained optimization problem

$$\text{minimize} \quad f(x) \quad \text{where} \quad x \in \mathcal{F} \tag{P}$$

for a functional $f \colon \mathbb{R}^n \to \mathbb{R}$ and a nonempty feasible set $\mathcal{F} \subseteq \mathbb{R}^n$. Further, define the penalized (unconstrained) problems

$$\text{minimize} \quad f(x) + \gamma \pi(x) \quad \text{where} \quad x \in \mathbb{R}^n \tag{P$_\gamma$}$$

for a penalty function $\pi \colon \mathbb{R}^n \to \mathbb{R}$ and a (penalty) parameter $\gamma > 0$.

Note: A penalty function is defined as satisfying $\pi(x) = 0$ for $x \in \mathcal{F}$ and $\pi(x) > 0$ for $x \in \mathbb{R}^n \setminus \mathcal{F}$.

Show the following:

(i) If $x^* \in \mathcal{F}$ is a local/global solution for (P$_\gamma$) for a $\gamma^* > 0$, then it is a local/global solution for (P) and for (P$_\gamma$) for any $\gamma \ge \gamma^*$.

(ii) If there exist a $\gamma^* > 0$ and an $x^* \in \mathbb{R}^n$, such that x^* is a global solution of (P$_\gamma$) for all $\gamma \ge \gamma^*$, then x^* is a global solution to (P).

(iii) Let f be differentiable. If $x^* \in \mathcal{F}$ is a local solution to (P) and to (P$_\gamma$) for a $\gamma^* > 0$, then π is not differentiable at x^* or $f'(x^*) = 0$.

What does Statement (*iii*) mean for exact penalization methods in general?

Homework Problem 13.4 (KKT conditions for convex multiobjective optimization) 6 Points

The filter-globalization strategy for the SQP method is based on ideas from multiobjective optimization, where the local/global minimizer concepts known from singleobjective optimization are replaced with so called Pareto-optimal points, i. e. points whose function value tuples are not dominated. This exercise gives a small insight into the field of multiobjective optimization.

Let $f: \mathbb{R}^n \to \mathbb{R}^m$ be a C^1-function whose components are convex.

(i) Show that the set $f(\mathbb{R}^n) + \mathbb{R}^m_{\geq 0}$ is convex.

(ii) Use the proper separation theorem for convex sets (e. g. Herzog, 2022, Satz 15.30) to show that all Pareto optimal points of the multiobjective optimization problem are global minimizers of corresponding single-objective optimization problems.

Note: Replacing multiobjective problems by corresponding single objective problems is known as scalarization.

(iii) Derive first order necessary Pareto-optimality conditions.

Please submit your solutions as a single pdf and an archive of programs via moodle.

References

Herzog, R. (2022). *Grundlagen der Optimierung*. Lecture notes. URL: https://tinyurl.com/scoop-gdo.