R. Herzog, G. Müller, E. Herberg, M. Hashemi
Heidelberg University

Exercise 12

Date issued: 3rd July 2023
Date due: 11th July 2023

Homework Problem 12.1 (Projected conjugate gradient method) 5 Points

Implement the projected M-preconditioned CG method Algorithm 13.2 and visualize the convergence behavior of the method

$$\begin{aligned} \text{Minimize} \quad & \frac{1}{2} d^\top A\, d - b^\top d \\ \text{subject to} \quad & B\, d = c \end{aligned}$$

for pseudo-randomized problem data (symmetric $A \in \mathbb{R}^{n \times n}$, $b \in \mathbb{R}^n$, $B \in \mathbb{R}^{m \times n}$ and $c \in \mathbb{R}^m$).

Homework Problem 12.2 (Generalized derivatives) 5 Points

(i) Compute the Bouligand- and Clarke generalized derivatives for $f: \mathbb{R} \to \mathbb{R}$, $f(x) = |x|$ at every $x \in \mathbb{R}$.

(ii) Show that if $f: \mathbb{R}^n \to \mathbb{R}^m$ is Lipschitz continuous on some neighborhood of $x \in \mathbb{R}^n$, then the Bouligand generalized derivative $\partial_B f(x)$ and the Clarke generalized derivative $\partial f(x)$ are nonempty and compact. In addition, $\partial f(x)$ is convex.

Homework Problem 12.3 (Semismooth NCP functions) 6 Points

Show that

$$\Phi_{\min}(a, b) := \min\{a, b\} \quad \text{“min” function,} \tag{13.8a}$$

$$\Phi_{\text{FB}}(a, b) := \sqrt{a^2 + b^2} - a - b \quad \text{Fischer-Burmeister function (Fischer, 1992)} \tag{13.8b}$$

as functions from $\mathbb{R}^2 \to \mathbb{R}$

(i) are NCP functions (Definition 13.4).

(ii) are semismooth everywhere (Definition 13.7).

Homework Problem 12.4 (Detecting convergence in primal-dual active set strategies) 6 Points

Consider the primal-dual active set strategy (semismooth Newton, Algorithm 13.10) for the lower bound constrained QP from the lecture notes with the iterates $(d^{(k)}, \mu^{(k)}, \lambda^{(k)})$, initialized with some $(d^{(0)}, \mu^{(0)}, \lambda^{(0)})$.

(i) Show that the residual $F(d^{(k)}, \mu^{(k)}, \lambda^{(k)})$ is nonzero only in its second component for $k \geq 1$.

(ii) Prove that when
$$\mathcal{A}(d^{(k)}, \mu^{(k)}) = \mathcal{A}(d^{(k+1)}, \mu^{(k+1)})$$
for some $k \in \mathbb{N}$ (the primal-dual active index sets coincide for two consecutive iterations) then $(d^{(k+1)}, \mu^{(k+1)}, \lambda^{(k+1)})$ is a solution of the constrained QP.

Please submit your solutions as a single pdf and an archive of programs via moodle.

References

Fischer, A. (1992). "A special Newton-type optimization method". *Optimization. A Journal of Mathematical Programming and Operations Research* 24.3-4, pp. 269–284. DOI: 10.1080/02331939208843795.