R. Herzog, G. Müller, E. Herberg, M. Hashemi
Heidelberg University

Exercise 11

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Homework Problem 11.1 (Examples for generalized Newton) 5 Points

For the nonlinear functions $F\colon \mathbb{R} \to \mathbb{R}$ and the set valued functions $N\colon \mathbb{R} \to \mathcal{P}(\mathbb{R})$ below, find all solutions z^* of the generalized equation

$$0 \in F(z) + N(z)$$

and determine, at which solutions the problem is strongly regular.

(i) $F(z) \coloneqq z^2 - 1$ and $N(z) \coloneqq \{0\}$

(ii) $F(z) \coloneqq z^2 - 1$ and $N(z) \coloneqq \mathbb{R}_{\geq}$

(iii) $F(z) \coloneqq (z-1)^2$ and $N(z) \coloneqq \mathcal{N}_{\mathbb{R}_{\geq}}(z)$

Homework Problem 11.2 (Generalized Newton for nonlinear complementarity problems) 5 Points

Implement the generalized Newton's method for solving inclusion problems of the form

$$0 \in F(z) + N(z),$$

where $F\colon \mathbb{R}^n \to \mathbb{R}^n$ is a function of class C^1 and $N\colon \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a set-valued map.

Make the termination criterion and the subproblem solver user-supplied parameters and apply your implementation to the problem in homework problem 11.1 item (iii). For the subsystem solves, you can employ a linear complementarity problem solver such as Lemke's method, see, e. g., https://github.com/AndyLamperski/lemkelcp and you can choose the termination criterion based on the violation of complementarity in terms of $\min(F(x), 0), \min(x, 0)$ and $|F(x)x|$.

Homework Problem 11.3 (Projected CG actually projects) 4 Points

Prove that Line 13 of the projected M-preconditioned CG method Algorithm 13.2 applied to the problem

$$\begin{aligned} \text{Minimize} \quad & \frac{1}{2} d^\top A\, d - b^\top d \\ \text{subject to} \quad & B\, d = c \end{aligned} \tag{13.1}$$

in fact computes the unique projection of the cost functionals M-steepest descent direction onto $\ker(B)$.

Homework Problem 11.4 (On the stopping criterion for the projected preconditioned CG) 6 Points

Consider the equality constrained linear-quadratic problem

$$\begin{aligned} \text{Minimize} \quad & f(d) := \frac{1}{2} d^\top A\, d - b^\top d \\ \text{subject to} \quad & B\, d = c \end{aligned} \tag{13.1}$$

where $A \in \mathbb{R}^{n \times n}$ is symmetric and $B \in \mathbb{R}^{m \times n}$ has full rank. Additionally, let $M \in \mathbb{R}^{n \times n}$ be s. p. d..

(i) Show that if d^* is a solution to (13.1), then $\operatorname{proj}_{\ker B}^M(M^{-1}\nabla f(d^*)) = 0$.

Hint: You can for example argue using convexity.

(ii) Let $Z \in \mathbb{R}^{n \times n-m}$ be an M-orthonormal matrix whose columns span the space $\ker(B)$, assume that $Z^\top A Z \in \mathbb{R}^{n-m \times n-m}$ is positive definite and let $d^* \in \mathbb{R}^n$ denote the unique solution to (13.1). Show that

$$\|d - d^*\|_A \leq \sqrt{\|(Z^\top A Z)^{-1}\|}\; \left\|\operatorname{proj}_{\ker B}^M(\nabla_M f(d))\right\|_M$$

for all $d \in \{d \in \mathbb{R}^n \mid Bd = c\}$.

Please submit your solutions as a single pdf and an archive of programs via moodle.