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Exercise 10

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Homework Problem 10.1 (Solvability and global solutions of equality constrained QPs) 6 Points

Prove Lemma 11.2 of the lecture notes, i. e., the following statements for the quadratic problem

$$\begin{aligned} &\text{Minimize} \quad \mathcal{L}(\bar{x}, \bar{\lambda}) + \mathcal{L}_x(\bar{x}, \bar{\lambda})\, d + \frac{1}{2} d^\intercal \mathcal{L}_{xx}(\bar{x}, \bar{\lambda})\, d, \quad \text{where } d \in \mathbb{R}^n \\ &\text{subject to} \quad h(\bar{x}) + h'(\bar{x})\, d = 0. \end{aligned} \tag{11.1}$$

(i) Suppose that the linear system $h(\bar{x}) + h'(\bar{x})\, d = 0$ is solvable, and that d_{part} is some particular solution. Suppose, moreover, that the reduced Hessian $Z^\intercal \mathcal{L}_{xx}(\bar{x}, \bar{\lambda})\, Z$ is positive semidefinite. Then the objective in the reduced QP (11.3) is convex. In this case, the following are equivalent:

(a) The QP (11.1) possesses at least one (global) minimizer.

(b) The QP (11.1) is neither unbounded nor infeasible.

(c) The KKT conditions (11.2) are solvable.

(d) The reduced QP (11.3) possesses at least one (global) minimizer.

(e) The reduced QP (11.3) is not unbounded.

(f) The first-order optimality condition (11.4) is solvable.

The global minimizers of (11.1) are precisely the KKT points, i. e., the d-components of solutions (d, λ) to the KKT system (11.2).

(ii) Suppose that the linear system $h(\overline{x}) + h'(\overline{x})\, d = 0$ is solvable, and that d_{part} is some particular solution. Suppose now that the reduced Hessian $Z^\intercal \mathcal{L}_{xx}(\overline{x}, \overline{\lambda})\, Z$ is not positive semidefinite. Then the QP (11.1) and the reduced QP (11.3) are unbounded.

(iii) Suppose that the linear system $h(\overline{x}) + h'(\overline{x})\, d = 0$ is not solvable. Then the QP (11.1) is infeasible and the reduced QP cannot be formulated for lack of a particular solution d_{part}.

Homework Problem 10.2 (LICQ is equivalent to a unique Lagrange multiplier for certain QPs) 3 Points

Consider the (affine linearly) equality constrained quadratic optimization problem of the form

$$\begin{aligned} \text{Minimize} \quad & \frac{1}{2} x^\intercal A x + b^\intercal x + c, \quad \text{where } x \in \mathbb{R}^n \\ \text{subject to} \quad & Cx = d \end{aligned} \tag{0.1}$$

for symmetric $A \in \mathbb{R}^{n \times n}$, $b \in \mathbb{R}^n$, $c \in \mathbb{R}$ and $C \in \mathbb{R}^{n \times n_{\text{eq}}}$, $d \in \mathbb{R}^{n_{\text{eq}}}$ and let x^* be a KKT-point of (0.1).

Show that the set $\Lambda(x^*)$ of Lagrange multipliers corresponding to x^* is a singleton if and only if the LICQ is satisfied at x^*.

Note: This proves the second set of equivalences in Equation (11.5).

Homework Problem 10.3 (Complementarity is equivalent to variational inequality) 3 Points

Prove Lemma 11.4 of the lecture notes, i. e. the equivalence of the KKT complementarity condition

$$\mu \ge 0, \quad g(x) \le 0, \quad \mu^\intercal g(x) = 0 \tag{11.11b}$$

and the variational inequality

$$\mu \in K \quad \text{and} \quad g(x)^\intercal (\nu - \mu) \le 0 \quad \text{for all } \nu \in K \tag{11.12}$$

with the closed convex cone $K := \mathbb{R}^{n_{\text{ineq}}}_{\ge 0}$ (the non-negative orthant).

Homework Problem 10.4 (On the normal cone) 3 Points

Prove Lemma 11.6 of the lecture notes, i. e., the following statements for a set $M \subseteq \mathbb{R}^n$ and $x \in M$:

(i) The normal cone is a closed convex cone.

(*ii*) $\mathcal{N}_M(x) = (M - \{x\})^\circ$ holds.

Additionally, prove that

(*iii*) $\mathcal{N}_M(x) \subseteq \mathcal{T}_M(x)^\circ$ but generally $\mathcal{N}_M(x) \neq \mathcal{T}_M(x)^\circ$.

Please submit your solutions as a single pdf and an archive of programs via moodle.