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Exercise 9

Date issued: 12th June 2023
Date due: 20th June 2023

Homework Problem 9.1 (Finding Solutions using First and Second Order Information) 6 Points

Consider the problem

$$\left.\begin{aligned} \text{Maximize} \quad & -(x_1-2)^2 - 2(x_2-1)^2 \quad \text{where } x \in \mathbb{R}^2 \\ \text{subject to} \quad & x_1 + 4x_2 \le 3 \\ \text{and} \quad & x_1 \ge x_2 \end{aligned}\right\}$$

Determine, which admissible points satisfy a constraint qualification (ACQ/GCQ/MFCQ/LICQ) and use first and second order information to compute all stationary points and solve the problem, i. e., find all optima and explain why they are local and/or global solutions.

Homework Problem 9.2 (Comparing the Strength of CQs) 6 Points

From the lecture notes, we know that

$$\boxed{\text{LICQ}} \xRightarrow{\text{Lemma 8.17}} \boxed{\text{MFCQ}} \xRightarrow{\text{Corollary 8.14}} \boxed{\text{ACQ}} \xRightarrow{\text{Definition 8.6}} \boxed{\text{GCQ}}. \tag{8.15}$$

Show that generally

$$\boxed{\text{LICQ}} \overset{(P3)}{\not\Leftarrow} \boxed{\text{MFCQ}} \overset{(P2)}{\not\Leftarrow} \boxed{\text{ACQ}} \overset{(P1)}{\not\Leftarrow} \boxed{\text{GCQ}}$$

by investigating the following problems P1 to P3 at $x^* = (0,0)^\intercal$:

$$\left.\begin{aligned} \text{Minimize} \quad & f(x) \qquad \text{where } x \in \mathbb{R}^2 \\ \text{subject to} \quad & x_1 \le 0 \\ & x_2 \le 0 \\ & x_1 x_2 = 0 \end{aligned}\right\} \tag{P1}$$

$$\left.\begin{aligned} &\text{Minimize} && f(x) && \text{where } x \in \mathbb{R}^2 \\ &\text{subject to} && q(x_1) - x_2 \le 0 \\ & && q(x_1) + x_2 \le 0 \end{aligned}\right\} \quad \text{for} \quad q(x_1) := \begin{cases} (x_1+1)^2, & x_1 < -1, \\ 0, & -1 \le x_1 \le 1, \\ (x_1-1)^2, & x_1 > 1, \end{cases} \tag{P2}$$

$$\left.\begin{aligned} &\text{Minimize} && f(x) && \text{where } x \in \mathbb{R}^2 \\ &\text{subject to} && -x_1^3 - x_2 \le 0 \\ & && -x_2 \le 0 \end{aligned}\right\} \tag{P3}$$

Homework Problem 9.3 (CQs are invariant under Slack Transformation) 10 Points

We can reformulate the original nonlinear problem

$$\left.\begin{aligned} &\text{Minimize} && f(x) && \text{where } x \in \mathbb{R}^n \\ &\text{subject to} && g_i(x) \le 0 && \text{for } i = 1, \ldots, n_{\text{ineq}} \\ &\text{and} && h_j(x) = 0 && \text{for } j = 1, \ldots, n_{\text{eq}} \end{aligned}\right\} \tag{7.1}$$

by introducing a so called **slack variable** $s \in \mathbb{R}^{n_{\text{ineq}}}$ to obtain the simple one-sided box-constrained problem

$$\left.\begin{aligned} &\text{Minimize} && f(x) && \text{where } (x, s) \in \mathbb{R}^{n \times n_{\text{ineq}}} \\ &\text{subject to} && g_i(x) + s = 0 && \text{for } i = 1, \ldots, n_{\text{ineq}} \\ &\text{and} && -s \le 0 \\ &\text{and} && h_j(x) = 0 && \text{for } j = 1, \ldots, n_{\text{eq}} \end{aligned}\right\}. \tag{7.1$_s$}$$

(*i*) Derive the KKT-system of (7.1$_s$) and show that there is a one-to-one connection between the solutions of the KKT systems corresponding to (7.1) and (7.1$_s$).

(*ii*) Show that GCQ/ACQ/MFCQ/LICQ is satisfied at a feasible (x, s) for (7.1$_s$) if the respective condition is satisfied at x for (7.1).

For which CQs can you show equivalence?

Homework Problem 9.4 (Multiplier Compactness is Equivalent to MFCQ) 6 Points (*i*) Use Farkas' Lemma (Lemma 7.11 in the lecture notes) to show that for $A \in \mathbb{R}^{p \times n}$ and $B \in \mathbb{R}^{n_{\text{eq}} \times n}$ with $\operatorname{rank}(B) = n_{\text{eq}}$ and $p \leq n_{\text{ineq}}$ either the system

$$Ad < 0, \quad Bd = 0 \tag{0.1}$$

has a solution $d \in \mathbb{R}^n$ or

$$A^\intercal \mu + B^\intercal \lambda = 0 \tag{0.2}$$

has a solution $(\mu, \lambda) \neq 0$ with $\mu \geq 0$.

Hint: Start with the existence of a nontrivial solution to (0.2). Focus the nontriviality on μ. Transform the conditions $\mu \neq 0, \mu \geq 0$ into a linear condition with a sign condition using a normalization step with respect to $\|\cdot\|_1$. Split λ into its positive and negative part. Apply Farkas' Lemma. Success.

(*ii*) Let (x^*, λ^*, μ^*) be a KKT-point of (7.1). Show that MFCQ is satisfied at x^* if and only if the set of Lagrange multipliers that solve the KKT system for x^* is compact.

Please submit your solutions as a single pdf and an archive of programs via moodle.