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Exercise 8

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Homework Problem 8.1 (Examples for Tangent-, Linearizing and Normal Cones) 5 Points

Consider the feasible set

$$F := \left\{x \in \mathbb{R}^n \,\middle|\, g_i(x) \le 0 \text{ for all } i = 1, \ldots, n_{\text{ineq}},\ h_j(x) = 0 \text{ for all } j = 1, \ldots, n_{\text{eq}}\right\} \tag{7.2}$$

without any equality restrictions h and with the inequality constraints $g\colon \mathbb{R}^3 \to \mathbb{R}^4$ defined by

$$g(x) = \begin{pmatrix} (x_1 - 1)^2 + x_2^2 - 1 \\ (x_1 - 3)^2 + x_2^2 - 1 \\ x_3 + 1 \\ -x_3 - 2 \end{pmatrix} \qquad \text{at} \qquad x^* = (2,\ 0,\ -1)^\intercal \in F.$$

Find the set of active indices $\mathcal{A}(x^*)$, an explicit representation of F, the tangent cone $\mathcal{T}_F(x^*)$, the **normal cone** $\mathcal{T}_F(x^*)^\circ$ and the linearizing cone $\mathcal{T}_F^{\text{lin}}(x^*)$ and sketch F and the cones.

Homework Problem 8.2 (Examples and Properties of Polar Cones) 4 Points

(i) Prove Lemma 7.9 of the lecture notes, i. e., for arbitrary sets $M, M_1, M_2 \subseteq \mathbb{R}^n$ the statements

- (a) M° is a closed convex cone.
- (b) $M_1 \subseteq M_2$ implies $M_2^\circ \subseteq M_1^\circ$.

(ii) Verify the claimed forms of the polar cones in Example 7.10, i. e., the following:

(a) Suppose that A is an affine subspace of $\mathbb{R}^n$ of the form $A = U + \{\bar{x}\}$. Then $A^\circ = \{\bar{x}\}^\circ \cap U^\perp$.

(b) In the absence of inequality constraints, the polar of the linearizing cone $\mathcal{T}_F^{\text{lin}}(x)$ for $x \in F$ has the representation

$$\begin{aligned}\mathcal{T}_F^{\text{lin}}(x)^\circ &= \{s \in \mathbb{R}^n \mid s \text{ is some linear combination of } h_j'(x)^\top,\ j = 1, \dots, n_{\text{eq}}\} \\ &= \operatorname{range} h'(x)^\top.\end{aligned}$$

(c) Let $N := (\mathbb{R}_{\ge 0})^n$ denote the non-negative orthant in $\mathbb{R}^n$. Then $N^\circ = (\mathbb{R}_{\le 0})^n$ is the non-positive orthant.

Homework Problem 8.3 (Lin. Cone and CQs Depend on Description of Feasible Set) 3 Points

The optimization problems

$$\begin{aligned}\text{Minimiere} \quad & f(x) \quad \text{über } x \in \mathbb{R} \\ \text{unter} \quad & x = 0\end{aligned} \tag{P_1}$$

and

$$\begin{aligned}\text{Minimiere} \quad & f(x) \quad \text{über } x \in \mathbb{R} \\ \text{unter} \quad & x^2 = 0\end{aligned} \tag{P_2}$$

for any $f \in C^1(\mathbb{R})$ have their obvious solution (because sole feasible point) at $x^* = 0$.

Show that the Abadie and Guignard constraint qualifications are satisfied at $x^* = 0$ for (P_1) but not (P_2).

Homework Problem 8.4 (ACQ for Problems with Affine Constraints) 6 Points

Consider

$$F := \left\{x \in \mathbb{R}^n \,\middle|\, g_i(x) \le 0 \text{ for all } i = 1, \dots, n_{\text{ineq}},\ h_j(x) = 0 \text{ for all } j = 1, \dots, n_{\text{eq}}\right\} \tag{7.2}$$

and

$$F^{\text{lin}}(x) = \left\{ y \in \mathbb{R}^n \,\middle|\, \begin{aligned} g_i(x) + g_i'(x)\,(y - x) &\le 0 \quad \text{for all } i = 1, \dots, n_{\text{ineq}} \\ h_j(x) + h_j'(x)\,(y - x) &= 0 \quad \text{for all } j = 1, \dots, n_{\text{eq}} \end{aligned} \right\}$$

for $x \in F$.

(i) Show that $\mathcal{T}_F^{\text{lin}}(x) = \mathcal{T}_{F^{\text{lin}}(x)}(x)$ for $x \in F$. (Remark 7.6 Statement (i))

(ii) Show that $\mathcal{T}_F^{\text{lin}}(x)$ is a closed convex cone. (Remark 7.6 Statement (ii))

(iii) Prove Theorem 8.9 by showing that the Abadie CQ holds at any feasible point of problems of the form

$$\left.\begin{aligned} \text{Minimize} \quad & f(x) && \text{where } x \in \mathbb{R}^n \\ \text{subject to} \quad & A_{\text{ineq}}\, x \le b_{\text{ineq}} \\ \text{and} \quad & A_{\text{eq}}\, x = b_{\text{eq}} \end{aligned}\right\} \tag{8.10}$$

Please submit your solutions as a single pdf and an archive of programs via moodle.