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Exercise 7

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Homework Problem 7.1 (Trust-Region Subproblem Solutions for varying Δ) 6 Points

Consider the trust region subproblems

$$\begin{aligned} &\text{Minimize} \quad q(s) = f(x) + f'(x)\,s + \frac{1}{2}\,s^\intercal H s, \quad \text{where } s \in \mathbb{R}^n \\ &\text{subject to} \quad \|s\|_M \le \Delta \end{aligned} \tag{6.1}$$

for $f \in \mathbb{C}^2$, a symmetric model Hessian $H \in \mathbb{R}^{n\times n}$, a Norm induced by the s. p. d. $M \in \mathbb{R}^{n\times n}$ and $\Delta > 0$ with the corresponding Δ family of subproblem solutions

$$S_{\bar\Delta} := \{s \in \mathbb{R}^n \mid s \text{ solves (6.1) for } \Delta \in [0, \bar\Delta]\}$$

and the corresponding necessary and sufficient optimality systems

$$\mu \ge 0, \quad \|s\|_M - \Delta \le 0, \quad \mu\,(\|s\|_M - \Delta) = 0 \tag{6.18a}$$

$$(H + \mu\,M)\,s = -f'(x)^\intercal \tag{6.18b}$$

$$H + \mu\,M \text{ is positive semidefinite} \tag{6.18c}$$

for $s \in \mathbb{R}^n$ and unique $\mu \in \mathbb{R}$.

(*i*) Derive a characterization for the solution s of (6.1) and the corresponding μ depending on Δ for the case where $H = M$ and interpret the result in terms of $S_{\bar\Delta}$.

(*ii*) Use (6.18) to create a visualization of S_2 for Rosenbrock's function for $M = \mathrm{Id}$, both choices of $x \in \{(0,-1)^\intercal, (0, 0.5)^\intercal\}$ and for both $H = \mathrm{Id}$ and $H = f''(x)$.

Homework Problem 7.2 (Saving a Function Evaluation per Trust-Region-Iteration) 3 Points

Remark 6.2 states that the quadratic model evaluation in the generic trust-region-Algorithm 6.1 is "typically a by-product of the solution of the trust-region subproblem (6.1)". Explain how the Steihaug-Toint-CG (Algorithm 6.14) has to be modified to be able to return this value for only three additional floating point operations per successfull CG iteration (plus minor additional work for initialization and termination).

Homework Problem 7.3 (Spektrum Shift for Symmetric Matrices) 3 Points

Let $H \in \mathbb{R}^{n\times n}$ be a symmetric matrix and $M \in \mathbb{R}^{n\times n}$ be s. p. d. Discuss for which values of $\mu \in \mathbb{R}$ the matrix $H + \mu M$ is positive definite, -semidefinite or indefinite, respectively.

Homework Problem 7.4 (Implementation of a Trust-Region-Method with Steihaug-CG) 6 Points

Implement a Newton-trust-region-method according to Algorithm 6.1 with Steihaug-Toint-CG (Algorithm 6.14). Keep a record of the trust region radius and the reduction ratio for visualization later. Apply your implementation to Rosenbrock's and/or Himmelblau's and discuss the results.

Please submit your solutions as a single pdf and an archive of programs via moodle.