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Exercise 3

Problem 3.1. (Energy minimization problem for a rope)

Many models in mechanics originate in energy minimization principles. In this exercise, we will consider the problem of determining the postion of a rope under gravitational forces in a discretized setting – i.e., a number N of weights of mass m_i at positions (x_i, y_i), $i = 1, \ldots, N$ that are connected by massless links. The links are assumed to be soft (but non-stretch) and the length of each link is at most ℓ.

(i) Model the problem for finding the unknown positions $(x, y) \in \mathbb{R}^N \times \mathbb{R}^N$ of the weights under gravitational load as a (finite-dimensional) potential energy minimization problem. At least one of the weights should be fixed at a certain position acting as supports, i.e.,

$$(x_i, y_i) = (\overline{x}_i, \overline{y}_i), \quad i \in L \subset \{1, \ldots, N\}, \, L \neq \emptyset.$$

Discuss the problems properties (linear/nonlinear, differentiable/non-smooth, convex/non-convex, etc.).

(ii) Can you expect LICQ to be satisfied at a minimizer (x^*, y^*)?

(iii) Formulate the KKT conditions of the problem for the nonsmooth and the squared, smooth link constraint formulation. Try to interpret the Lagrange multipliers as physical quantities. Check what will happen to the multipliers when you take the limit $N \to \infty$ while keeping the total weight and length of the chain constant in either formulation.

(iv) Suppose that the positions of the weights are constrained (in addition to the constraints already present due to the links and supports) by a rigid obstacle. The space occupied by the obstacle is described by $\{(x, y) \in \mathbb{R}^2 : \phi(x, y) \geq 0\}$, leading to the non-penetration conditions $\phi(x_i, y_i) \leq 0$ for $i = 1, \ldots, N$.

How do the KKT conditions change?

(*v*) Solve the obstacle-constrained problem for some provided test configurations and investigate the behavior.