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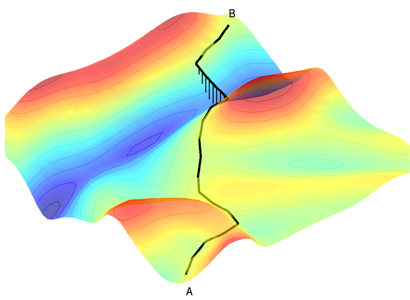
Exercise 1

Problem 1.1. (Railway planning problem)

Given two points A and B in $\mathbb{R}^3$ (that are reasonably close), we are looking to construct a railway connecting A and B at the lowest cost possible. The topography of the region around A and B is described by a function $z: \mathbb{R}^2 \to \mathbb{R}$ mapping a point $(x, y) \in \mathbb{R}^2$ to the vertical height above sea level $z(x, y)$ at (x, y), i. e., the points on the surface of the earth are described by $(x, y, z(x, y))$. We need to avoid steep inclines along the new railway, a requirement that we simplify by assuming that A and B are at sea level ($A_z = B_z = 0$) and requiring that the entire railway remains at sea level as well. To achieve this, we may build tunnels and bridges through/over topographic features, both of which cost money depending on their depth/height. The total cost for a piece of railway of infinitesimal length ds at the position $(x, y, z(x, y))$ is given by

$$(C_1 + C_2\, z(x, y)^2)\ \mathrm{d}s$$

for constants $C_1, C_2 > 0$.



Topography around A and B with tunnels and bridges.

Model the problem as an optimization problem of the form (1.1) in the lecture notes that can be solved numerically and approximately solve the problem for the parameters

- $A = (-1, -1, 0)^\top$, $B = (1, 1, 0)^\top$
- $z(x, y) = \frac{1}{8} \sum_{i=1}^{6} a_i(x, y)$ with

$$
\begin{aligned}
a_1(x, y) &= -f(3 - 0.7x^2 - 5y) \\
a_2(x, y) &= f(5(x - 0.7)^2 + 20(y - 0.1)^2) \\
a_3(x, y) &= f(4 + 2x^2 - 4y) \\
a_4(x, y) &= 0.5 \sin(4(x - y^2)) \sin(3(y + x^2)) \\
a_5(x, y) &= 0.6 f(50((x + 1)^2 + (y + 0.8)^2 - 0.25)^2) \\
a_6(x, y) &= -1/6
\end{aligned}
$$

  where $f(x) = 2/(1 + x^2)$.

using a suitable numerical solver of your choice. Investigate the effects of the parameters C and the initial guess that you supply to the solver.